Toulouse School of Economics, 2009-2010
Macroeconomics II – Franck Portier

Homework 2
Solutions

Problem I – A Simplified Real-Business-Cycle Model with Additive Technology Shocks

Consider an economy consisting of a constant population of infinitely-lived individuals. The representative individual maximizes the expected value of $\sum_{t=0}^{\infty} u(C_t)/(1+\rho)^t$, $\rho > 0$. The instantaneous utility function, $u(C_t)$, is $u(C_t) = C_t - \theta C_t^2$, $\theta > 0$. Assume that C is always in the range where $u'(C)$ is positive.

Output is linear in capital, plus an additive disturbance: $Y_t = AK_t + e_t$. There is no depreciation; thus $K_{t+1} = K_t + Y_t - C_t$, and the interest rate is A. Assume $A = \rho$. Finally, the disturbance follows a first-order autoregressive process: $e_t = \phi e_{t-1} + \varepsilon_t$, where $-1 < \phi < 1$ and where the ε_t's are mean-zero, i.i.d. shocks.

1 – Find the first-order condition (Euler equation) relating C_t and expectations of C_{t+1}.

The Lagrangian of the Social Planner problem is

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \left(\frac{1}{1-\rho}\right)^t \left(C_t - \theta C_t^2 + \lambda_t \left(C_t + K_{t+1} - (1+1)K_t + e_t\right)\right)$$

FOC with respect to C_t and K_{t+} are:

$$C_t = E_t C_{t+1} \tag{1}$$

$$\lambda_t = E_t \frac{A+1}{1+\rho} \lambda_{t+1} \tag{2}$$

The Euler equation for consumption then writes

$$C_t = E_t C_{t+1}$$

2 – Guess that consumption takes the form $C_t = \alpha + \beta K_t + \gamma e_t$. Given this guess, what is K_{t+1} as a function of K_t and e_t?

The resource constraint implies $K_{t+1} = (A+1)K_t - C_t + e_t$. Replacing C_t by its expression $\alpha + \beta K_t + \gamma e_t$, one gets

$$K_{t+1} = (A+1-\beta)K_t - \alpha + (1-\gamma)e_t \tag{$\star$}$$

3 – What values must the parameters α, β, and γ have for the first-order condition in part (1) to be satisfied for all values of K_t and e_t?

Let's compute $E_t C_{t+1}$ using equation ($\star$):

$$\begin{aligned} E_t C_{t+1} &= \alpha + \beta E_t K_{t+1} + \gamma E_t e_{t+1} \\ &= \alpha(1-\beta) + \beta(A+1-\beta)K_t + (\beta(1-\gamma) + \gamma\phi)e_t \end{aligned}$$

The Euler equation implies $C_t = E_t C_{t+1}$. Knowing that $C_t = \alpha + \beta K_t + \gamma e_t$, one obtains the conitions

$$\begin{aligned} \alpha &= \alpha(1-\beta) \\ \beta &= \beta(A+1-\beta) \\ \gamma &= \beta(1-\gamma) + \gamma\phi \end{aligned}$$

from which we obtain

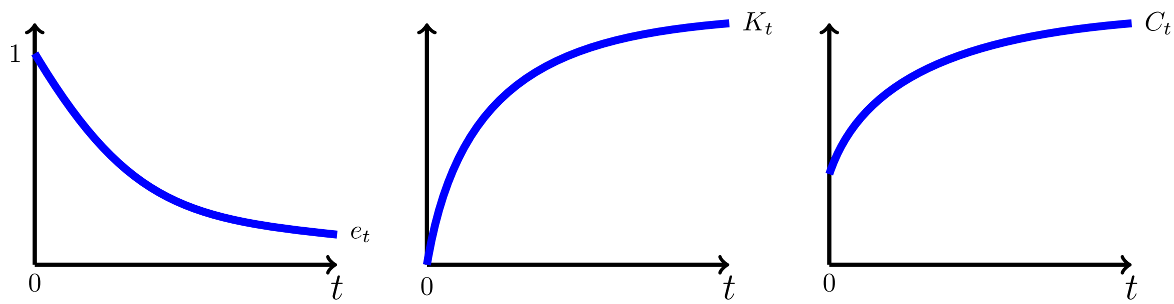
$$\begin{aligned} \alpha &= 0 \\ \beta &= A \\ \gamma &= \frac{A}{A+1-\phi} \end{aligned}$$

The equilibrium then writes

$$K_{t+1} = K_t + \frac{1-\phi}{A+1-\phi} e_t$$
$$C_t = AK_t + \frac{A}{A+1-\phi} e_t$$

4 – What are the effects of a one-time shock to ε on the paths of Y, K, and C?

Figure 1: Impulse response to a one time shock ε



PROBLEM II – AN ANALYTIC MODEL WITH LOG-LINEAR DEPRECIATION

Consider a model economy populated with a representative household and a representative firm. The firm has a Cobb-Douglas technology:

$$Y_t = Z_t K_t^{\gamma} N_t^{1-\gamma} \tag{3}$$

where K_t is capital, N_t labor input, and Z_t a stochastic technological shock. All profits of the firm are distributed to the household. Capital evolves according to the log linear relation

$$K_{t+1} = AK_t^{1-\delta} I_t^{\delta} \qquad 0 < \delta \leq 1 \tag{4}$$

where δ is the rate of depreciation and where I_t is investment in period t.

The representative household works N_t, consumes C_t in period t, and ends the period with a quantity of money M_t.

$$U = E_0 \sum_{t=0}^{\infty} \beta^t \left[\log C_t + \omega \log \frac{M_t}{P_t} - V(N_t) \right] \tag{5}$$

where V is a convex function. At the beginning of period t there is a stochastic multiplicative monetary shock denoted by μ_t. The money holdings M_{t-1} carried from the previous period are multiplied by μ_t, so that the household starts period t with money holdings $\mu_t M_{t-1}$. Capital is accumulated by the household and rented to the firm.

Let κ denote the real rental rate of capital, P the price of the final good and W the nominal wage.

We first assume that $\delta = 1$ (full depreciation).

1 – Write down the budget constraint of the household and the profit function of the firm

2 – Derive FOCs of the utility and profit maximization

3 – Define a competitive equilibrium of this economy

4 – Solve the model and show that N_t is constant along an equilibrium path.

5 – Derive a $AR(1)$ process for log of output. Draw the Impulse Response Function to a unit shock to z and μ, assuming that both shocks are *iid*. What are the determinants of the' size and persistence of those IRFs?

⋆

We assume now that $\delta \in]0,1[$.

6 – What is the economic meaning of equation (2)?

7 – We still denote κ_t the real return in t on investment I_{t-1}:

$$\kappa_t = \frac{\partial Y_t}{\partial I_{t-1}} \tag{6}$$

and note that the price, in terms of current output, of capital to be transmitted to the next period is not one as when $\delta = 1$. We denote it as q_t (like Tobin's q), and it is equal to:

$$q_t = \frac{\partial K_{t+1}/\partial K_t}{\partial K_{t+1}/\partial I_t} \tag{7}$$

8 – Comment the new budget constraint of the household

$$C_t + \frac{M_t}{P_t} + I_t = \frac{W_t}{P_t}N_t + \kappa_t(I_{t-1} + q_{t-1}K_{t-1}) + \frac{\mu_t M_{t-1}}{P_t} \tag{8}$$

9 – Solve for the competitive equilibrium of this economy and compute $\frac{I_t}{C_t}$ and $\frac{M_t}{P_t Y_t}$

10 – Compute the constant level of N.

11 – Derive the new process for log of output. Draw typical IRFs and discuss of their shape as a function of parameters.

The first part of the exercise has been extensively done in the course. In each period the firm demands labor competitively so that the real wage is equal to the marginal productivity of labor:

$$\frac{W_t}{P_t} = \frac{\partial Y_t}{\partial N_t} = (1-\gamma)\frac{Y_t}{N_t} \tag{9}$$

Also the real return on capital is simply the marginal productivity of capital :

$$\kappa_t = \frac{\partial Y_t}{\partial K_t} = \gamma \frac{Y_t}{K_t} \tag{10}$$

The household maximizes the expected value of his discounted utility subject to the sequence of budget constraints. Call λ_t the marginal utility of real wealth in period t (i.e. the Lagrange multiplier associated with the corresponding budget constraint. Then the usual optimality conditions for the consumer's program yield:

$$\frac{1}{C_t} = \lambda_t \tag{11}$$

$$V'(N_t) = \lambda_t \frac{W_t}{P_t} \tag{12}$$

$$\lambda_t = \beta E_t(\lambda_{t+1}\kappa_{t+1}), \tag{13}$$

$$\lambda_t = \frac{\omega P_t}{M_t} + \beta E_t\left(\lambda_{t+1}\frac{\mu_{t+1}P_t}{P_{t+1}}\right) \tag{14}$$

Combining (11), (13), the condition $Y_t = C_t + I_t$ and the definition of κ_t in (10), we obtain:

$$\frac{I_t}{C_t} = \beta\gamma + \beta\gamma E_t\left(\frac{I_{t+1}}{C_{t+1}}\right) \tag{15}$$

which, by solving forward, yields :

$$\frac{I_t}{C_t} = \frac{\beta\gamma}{1-\beta\gamma} \tag{16}$$

so that:

$$C_t = (1-\beta\gamma)Y_t \tag{17}$$

$$I_t = K_{t+1} = \beta\gamma Y_t \tag{18}$$

The equilibrium condition for money is that the quantity of money demanded by the household, M_t, be equal to the initial money holdings $\mu_t M_{t-1}$:

$$M_t = \mu_t M_{t-1} \tag{19}$$

Now condition (14), using (11) and (19), is rewritten as :

$$\frac{M_t}{P_t C_t} = \omega + \beta E_t \left(\frac{M_{t+1}}{P_{t+1} C_{t+1}} \right) \tag{20}$$

which solves as:

$$\frac{M_t}{P_t C_t} = \frac{\omega}{1-\beta} \tag{21}$$

Combining (17) and (21), we obtain the level of real money balances :

$$\frac{M_t}{P_t} = \frac{\omega(1-\beta\gamma)}{1-\beta} Y_t = \varrho Y_t \tag{22}$$

Now combining condition (12) with the expression of the real wage (9) and the value of consumption (17), we find that N_t is constant and equal to N, where N is given by:

$$N V'(N) = \frac{1-\gamma}{1-\beta\gamma} \tag{23}$$

Then formula (23) yields a Walrasian quantity of labor equal to:

$$N = \left[\frac{1-\gamma}{\xi\,(1-\beta\gamma)} \right]^{1/\nu} \tag{24}$$

To sum up, the dynamics is given by

$$N_t = N \tag{25}$$

$$Y_t = Z_t K_t^{\gamma} N^{1-\gamma} \tag{26}$$

$$\frac{W_t}{P_t} = (1-\gamma) \frac{Y_t}{N} \tag{27}$$

$$K_{t+1} = \beta\gamma Y_t \tag{28}$$

Consider now the case where capital evolves according to the log linear relation

$$K_{t+1} = A K_t^{1-\delta} I_t^{\delta} \qquad 0 < \delta \leq 1 \tag{29}$$

where δ is the rate of depreciation. The case studied in the main text corresponds to $\delta = 1$ and $A = 1$. We shall now see how this modifies the analysis. A first thing to note is that the price, in terms of current output, of capital to be transmitted to the next period is not one. We denote it as q_t (like Tobin's q), and it is equal to:

$$q_t = \frac{\partial K_{t+1} / \partial K_t}{\partial K_{t+1} / \partial I_t} = \frac{1-\delta}{\delta} \frac{I_t}{K_t} \tag{30}$$

We still call κ_t the real return in t on investment I_{t-1}. We have:

$$\kappa_t = \frac{\partial Y_t}{\partial I_{t-1}} = \frac{\partial Y_t}{\partial K_t} \frac{\partial K_t}{\partial I_{t-1}} = \gamma\delta \frac{Y_t}{I_{t-1}} \tag{31}$$

The household's budget constraint is now replaced by

$$C_t + \frac{M_t}{P_t} + I_t = \frac{W_t}{P_t} N_t + \kappa_t (I_{t-1} + q_{t-1} K_{t-1}) + \frac{\mu_t M_{t-1}}{P_t} \tag{32}$$

The household maximizes his utility:

$$\sum_t \beta^t \left[\log C_t + \omega \log \frac{M_t}{P_t} - V(L_t) \right] \tag{33}$$

subject to his budget constraints and the capital accumulation equations. The Lagrangean for this program is:

$$\sum_t \beta^t \left[\log C_t + \omega \log \frac{M_t}{P_t} - V\left(L_t\right)\right] + \sum_t \beta^t \lambda_t \left[\frac{W_t}{P_t} N_t + \kappa_t (I_{t-1} + q_{t-1} K_{t-1}) + \frac{\mu_t M_{t-1}}{P_t} - C_t - I_t - \frac{M_t}{P_t}\right] \tag{34}$$

$$+ \sum_t \beta^t \zeta_t \left(A K_{t-1}^{1-\delta} I_{t-1}^{\delta} - K_t\right)$$

The first order conditions for investment and capital yield:

$$\lambda_t = \beta E_t \left(\lambda_{t+1} \kappa_{t+1}\right) + \beta \delta E_t \left(\frac{\zeta_{t+1} K_{t+1}}{I_t}\right) \tag{35}$$

$$\zeta_t = \beta E_t \left(\lambda_{t+1} \kappa_{t+1} q_t\right) + \beta \left(1 - \delta\right) E_t \left(\frac{\zeta_{t+1} K_{t+1}}{K_t}\right) \tag{36}$$

Comparing the two, and using the definition of q_t, we find:

$$\zeta_t = \frac{1-\delta}{\delta} \frac{I_t}{K_t} \lambda_t \tag{37}$$

Inserting this into either (35) or (36) we obtain:

$$\frac{I_t}{C_t} = \beta \gamma \delta E_t \left(\frac{Y_{t+1}}{C_{t+1}}\right) + \beta \left(1 - \delta\right) E_t \left(\frac{I_{t+1}}{C_{t+1}}\right) \tag{38}$$

$$= \beta \gamma \delta + \left[\beta \gamma \delta + \beta \left(1 - \delta\right)\right] E_t \left(\frac{I_{t+1}}{C_{t+1}}\right) \tag{39}$$

which solves as:

$$\frac{I_t}{C_t} = \frac{\beta \gamma \delta}{1 - \beta \left(1 - \delta + \gamma \delta\right)} \tag{40}$$

so that:

$$C_t = \frac{1 - \beta \left(1 - \delta + \gamma \delta\right)}{1 - \beta + \beta \delta} Y_t \tag{41}$$

$$I_t = \frac{\beta \gamma \delta}{1 - \beta + \beta \delta} Y_t \tag{42}$$

Now equation (21) is still valid, and combining it with (41), we obtain:

$$\frac{M_t}{P_t} = \frac{\omega \left[1 - \beta \left(1 - \delta + \gamma \delta\right)\right]}{\left(1 - \beta\right) \left(1 - \beta + \beta \delta\right)} Y_t. \tag{43}$$

Finally, equations (9), (11), (12) and (41) yield a Walrasian quantity of labor N now given by :

$$N \; V'(N) = \frac{\left(1 - \gamma\right) \left(1 - \beta + \beta \delta\right)}{1 - \beta + \beta \delta - \beta \gamma \delta} \tag{44}$$

Let us now move to the situation with wage contracts and denote as σ the savings rate, so that (42) is rewritten:

$$I_t = \sigma Y_t \qquad \sigma = \frac{\beta \gamma \delta}{1 - \beta + \beta \delta} \tag{45}$$

Combining (45) with equation (29) and the production function, we find that output is given by:

$$y_t = \frac{[1 - (1 - \delta) L][z_t + (1 - \gamma) n_t]}{1 - (1 - \delta + \gamma \delta) L} + \frac{\gamma \delta \log \sigma + \gamma \log A}{\delta (1 - \gamma)} \tag{46}$$

Moreover employment is still given by:

$$n_t = n + \varepsilon_{mt} \tag{47}$$

Combining (46) and (47) we obtain the final expression for output:

$$y_t = n + \frac{[1 - (1 - \delta) L][z_t + (1 - \gamma) \varepsilon_{mt}]}{1 - (1 - \delta + \gamma \delta) L} + \frac{\gamma \delta \log \sigma + \gamma \log A}{\delta (1 - \gamma)} \tag{48}$$