Chapter 4

Monetary Business Cycles

- Main Reference:
  - Jordi Galì, Monetary Policy, Inflation and the Business Cycle, PUP, 2008

- Other references that could be read :
  - Carl Walsh, Monetary Theory and Practice, MIT Press 1998, chapters 1-3
  - George McCandless and Warren Weber, Some Monetary Facts, Minneapolis Fed QR, 1995

1 Introduction

- What do we know about the effect of money and monetary policy?
- How can we understand the existence of a positive price for money in GE?
- What do we know about optimality of monetary policy?

2 "Facts"

2.1 Long run facts

- McCandless and Weber: 110 countries over 30 years
- They compute the long-run geometric average rate of growth for:

  - the standard measure of production : gross domestic product adjusted for inflation (real GDP);

  - a standard measure of the general price level: consumer prices;

  - three commonly used definitions of money (M0, M1, and M2)

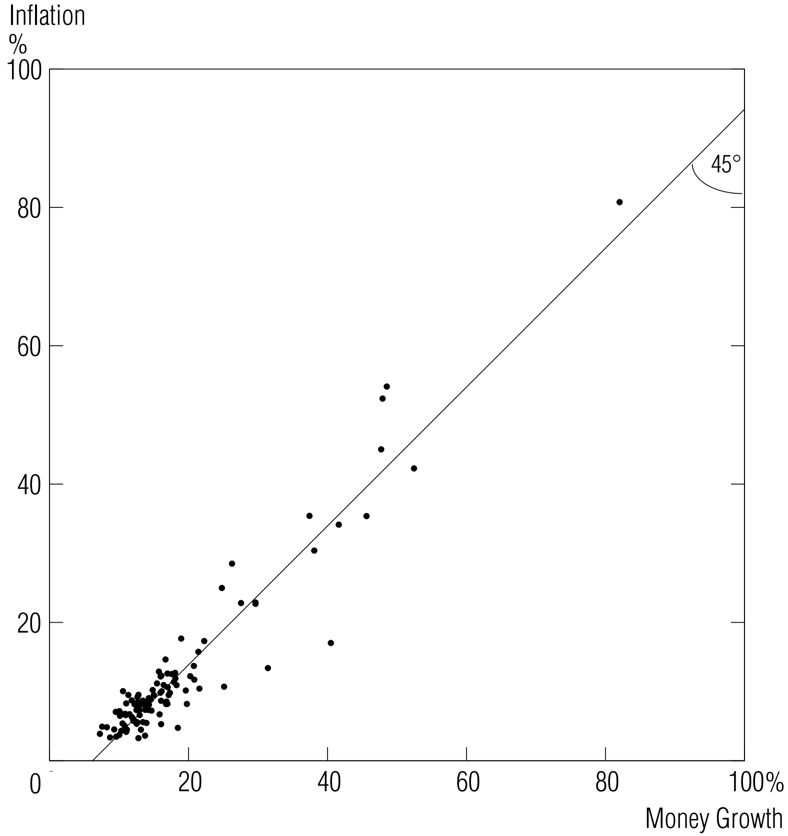
- They also look at 2 more homogenous sub samples: 21 OECD countries and 14 Latin American countries.
- Three main results:

RESULT 1 **Money Growth and Inflation :** *In the long run, there is a high (almost unity) correlation between the rate of growth of the money supply and the rate of inflation. This holds across three definitions of money and across the full sample of countries and two subsamples.*

Chart 1

Money Growth and Inflation: A High, Positive Correlation

Average Annual Rates of Growth in M2 and in Consumer Prices During 1960–90 in 110 Countries



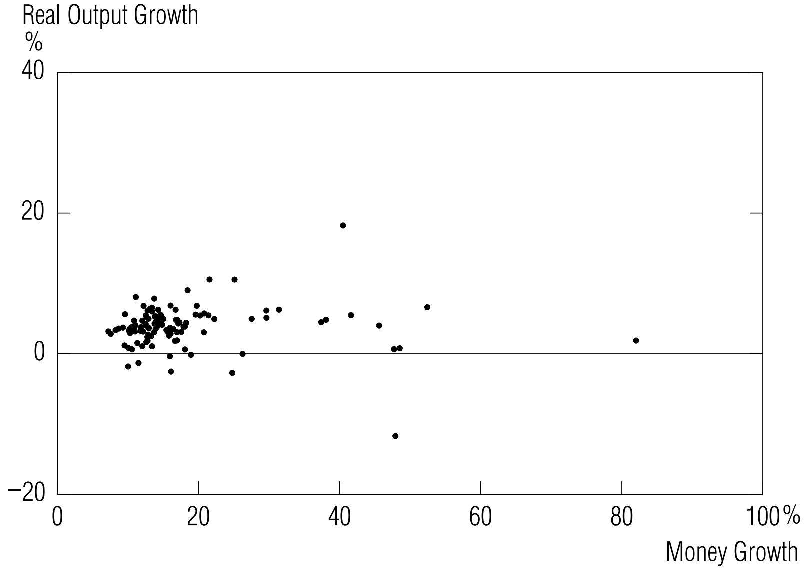
Source: International Monetary Fund

RESULT 2 **Money Growth and Real Output Growth :**
In the long run, there is no correlation between the growth rates of money and real output. This holds across all definitions of money, but not for a subsample of OECD countries, where the correlation seems to be positive.

Chart 2

Money and Real Output Growth: No Correlation in the Full Sample . . .

Average Annual Rates of Growth in M2
and in Nominal Gross Domestic Product, Deflated by Consumer Prices
During 1960–90 in 110 Countries

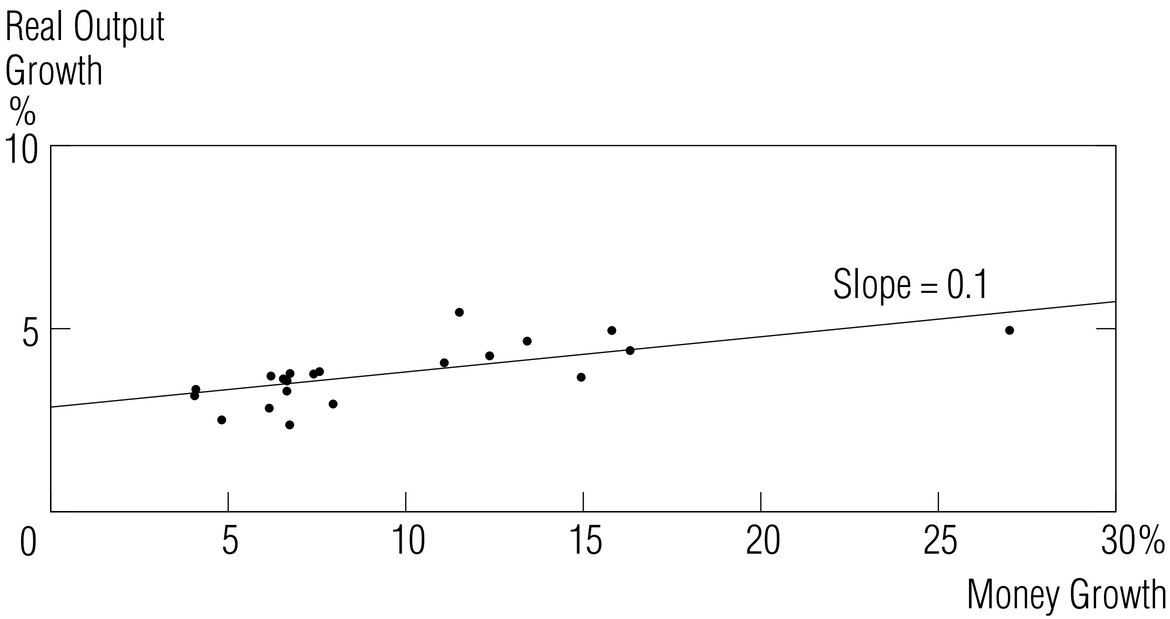


Source: International Monetary Fund

Chart 3

. . . But a Positive Correlation in the OECD Subsample

Average Annual Rates of Growth in M0
and in Nominal Gross Domestic Product, Deflated by Consumer Prices
During 1960–90 in 21 Countries



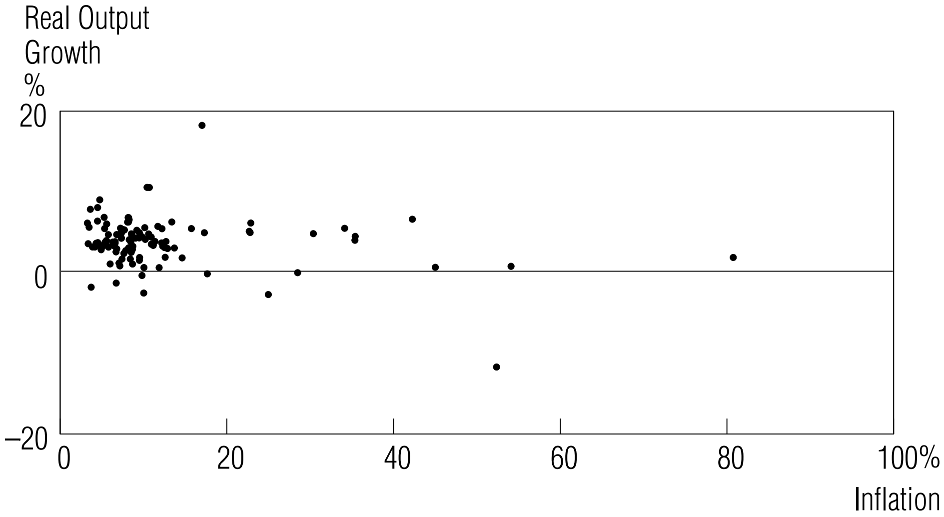
Source: International Monetary Fund

RESULT 3 **Inflation and Real Output Growth :** *In the long run, there is no correlation between inflation and real output growth. This finding holds across the full sample and both subsamples.*

Chart 4

Inflation and Real Output Growth: No Correlation

Average Annual Rates of Growth in Consumer Prices
and in Nominal Gross Domestic Product, Deflated by Consumer Prices
During 1960–90 in 110 Countries



Source: International Monetary Fund

2.2 Short Run Evidence

- It is much more difficult to get non controversial results in the short run, because money is partly endogenous.
- Money and output may vary because money causes output, but also because output causes money (banking system), or as a common response to a third variable (technology)

2.2.1 Traditional approach

- Unconditional correlations suggest that the correlation is positive and that money is leading
- The correlation is larger for broader aggregate like M2, which is more endogenous.
- Seminal work of Friedman and Schwartz (1963): money matters for BC because it leads output
- But Tobin (1970): *post hoc ergo propter hoc* (reverse causality)

$\leadsto$ need for conditional correlations.

- On top of that, monetary aggregates need not to be always the instrument of the monetary authorities (money market interest rate is now the most commonly used instrument)
- Sims (1972): Does money Granger-cause output:

$$y_t = y_0 + \sum_i a_i m_{t-1} + \sum_i b_i y_{t-1} + \sum_i c_i z_{t-1} + e_t$$

- If all a_i are zeros, then money does not cause output. $\leadsto$ evidence are that it does, but results depends on the specification (lags, extra variables z, detrending, etc...)

- Extra work by Barro (1978): does both anticipated and unanticipated parts of money matter fore real output $\leadsto$ only the unanticipated one for Barro (but some contradictive findings later)

2.2.2 Monetary VAR's

- The idea here is to "purge" the monetary policy instrument from responses to other shocks, so that the "pure" response to a monetary shocks can be estimated.
- Here I consider the estimation of Christiano, Eichenbaum and Evans in the Handbook of Macro.
- Starting point: assumption that the Central Bank implements its monetary policy using a reaction function.

- At each period t, the policymaker sets its instrument, S_t, in a systematic way, in relation with the information set of the period, Ω_t.
- The monetary policy rule can be written:

$$S_t = f\left(\Omega_t\right) + \sigma_s \epsilon_s^t \tag{1}$$

where S_t is the instrument of the Central Bank and $f(.)$ is a (approximatively) linear function that relates the instrument to the information set of the central bank Ω_t.

- ϵ_s^t defines the monetary policy shocks, with standard deviation σ_S.

- Assume that the variables of interest - including the instruments of the monetary policy - $\{Z_t, t = 1, ...T\}$ follow a VAR of order q:

$$A_0 Z_t = A_1 Z_{t-1} + ...A_q Z_{t-q} + \epsilon_t \qquad (2)$$

- The structural shocks ϵ_t, which include the monetary policy shocks, are by construction orthogonal one to each others.
- A_0, the matrix of contemporaneous impact, is assumed to be invertible.

• Pre-multiplying (2) by A_0^-1, one obtains the reduced VAR representation:

$$Z_t = B_1 Z_{t-1} + ... B_q Z_{t-q} + u_t \tag{3}$$

where $B(L) = A_0^{-1} A(L)$ and $u_t = A_0^{-1} \epsilon_t$ has covariance matrix V.

• u_t are uncorrelated with lagged values of Z_t, and the parameters $\{B_1, ..., B_q, V\}$ can be estimated consistently using Ordinary Least Squares method.

• The whole problem is to go from u to ϵ. To do so, the matrix A_0 is needed.

• Assuming that the covariance matrix of ϵ is identity (by normalization), the definition of u gives

$$V = A_0^{-1}\left(A_0^{-1}\right)'$$

• This gives us only $(n \times (n+1))/2$ restrictions (V is symmetric) $\leadsto$ we need some identifying assumptions.

• Assume that the matrix of contemporaneous impacts A_0 is lower triangular.

• Then one can recover A_0^{-1} by the Cholevsky decomposition of

V.

- Therefore, one has a recursive system which depends on the order of the variables in Z_t.

- Let's decompose Z_t in three groups, according to the location of the instrument S_t:

$$Z_t = \begin{pmatrix} X_t^1 \\ S_t \\ X_t^2 \end{pmatrix}$$

- The recursiveness assumption implies that when the central bank sets its instrument S_t, it does not observe the contemporaneous values X_t^1, but does observe X_t^2.
- Another implication is that the variables in X_t^1 react to a monetary policy shock with a delay of one period.

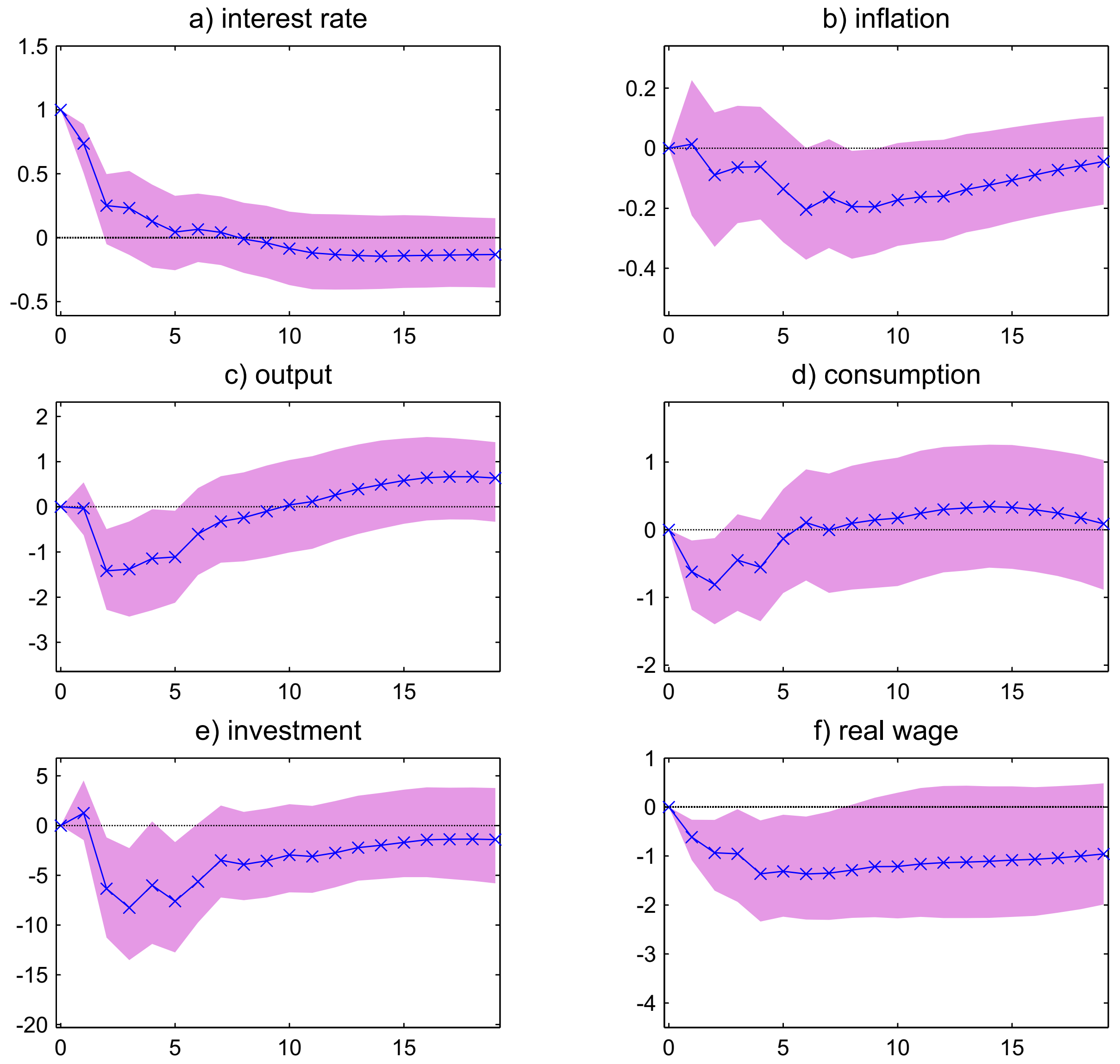
- In CEE paper (US economy):

  - the instrument used by the central bank is the three months federal fund rate;

  -X_t^1 is composed of GDP, GDP deflator and an Index of Crude Good price;

  - X_t^2 regroups the variables actually used by the Federal Reserve bank to control the level of the Federal Fund rate: Non Borrowed Reserve, Total Reserve and M1.

- After the estimation of all the parameters of the structural VAR, one can compute impulse response functions to a monetary policy shock.


a) interest rate
b) inflation
c) output
d) consumption
e) investment
f) real wage

- Results: Not much response of prices in the short run, expansionary effect, liquidity effect $\rightsquigarrow$ short run non neutrality

3 A Classical Monetary Model

3.1 Households

- Objective :

$$E_0 \sum_{t=0}^{\infty} \beta^t U\left(C_t, N_t, \frac{M_t}{P_t}\right)$$

- Budget Constraint :

$$P_t C_t + M_t + Q_t B_t \leq B_{t-1} + W_t N_t + M_{t-1} + T_t$$

- T_t gathers lump sum transfers (money, taxes) and dividends
- B_t is a riskless one-period bond that pays 1 unit of good in $t+1$
- $\mathcal{A}_t = B_{t-1} + M_{t-1}$, $\lim_{T\to\infty} E_t \mathcal{A}_T \geq 0$

3.1.1 Optimal C, N and M

- FOC:

$$\begin{cases} -\frac{U_{n,t}}{U_{c,t}} = \frac{W_t}{P_t} \\ Q_t = \beta E_t \left[\frac{U_{c,t+1}}{U_{c,t}} \frac{P_t}{P_{t+1}} \right] \\ \frac{U_{m,t}}{U_{c,t}} = 1 - Q_t \end{cases}$$

- We assume in the following: $U = \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1-\phi}}{1-\phi} + \frac{\left(\frac{M_t}{P_t}\right)^{1-\nu}}{1-\nu}$

- Then the FOC become

$$\begin{cases} \frac{W_t}{P_t} = C_t^{\sigma} N_t^{\phi} & (a) \\ \frac{M_t}{P_t} = C_t^{\sigma/\nu} (1 - Q_t)^{-1/\nu} & (b) \\ 1 = \beta E_t \left[\frac{1}{Q_t} \left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{P_t}{P_{t+1}} \right] & (c) \end{cases}$$

- We log-linearize those equations (some are already loglinear, we just need to take logs).
- We use the notation $x = \log X$
- (a) gives $w_t - p_t = \sigma c_t + \phi n_t$

3.1.2 Linearization of Equation (c)

- Define $i_t = \log \frac{1}{Q_t} = -\log Q_t$: nominal interest rate ($Q_t = \exp(-i_t)$))
- Define $\pi_{t+1} = p_{t+1} - p_t = \log\left(\frac{P_{t+1}}{P_t}\right)$: inflation
- Let $\rho = \log \frac{1}{\beta}$

- Take (c) and replace X by $\exp(\log(X)) = \exp(x)$

$$1 = \beta E_t \left[\frac{1}{Q_t} \left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{P_t}{P_{t+1}} \right]$$

$$\iff 1 = E_t \left[\exp \left\{ \log \left(\frac{1}{Q_t} \right) - \sigma \log \left(\frac{C_{t+1}}{C_t} \right) - \log \left(\frac{P_{t+1}}{P_t} \right) + \log(\beta) \right\} \right]$$

$$\iff 1 = E_t \left[\exp \left\{ i_t - \sigma \Delta c_{t+1} - \pi_{t+1} - \rho \right\} \right]$$

- Consider the non stochastic perfect foresight steady state with constant inflation π and constant real growth γ.

- (c) implies $1 = \exp(i - \sigma\gamma - \pi - \rho)$
- and therefore $i = \rho + \pi + \sigma\gamma$

How to take a first order expansion of $\exp(x)$ around 0?

$$\exp(x_t) \approx \exp(0) + \exp(0) \times (x_t - 0) = 1 + x_t$$

- Here:

$$\begin{aligned}\exp(i_t - \sigma\Delta c_{t+1} - \pi_{t+1} - \rho) &\approx 1 + (i_t - i) - \sigma(\Delta c_{t+1} - \gamma) - (\pi_{t+1} - \pi) \\ &\approx 1 + i_t - \sigma\Delta c_{t+1} - \pi_{t+1} - (i - \sigma\gamma - \pi) \\ &\approx 1 + i_t - \sigma\Delta c_{t+1} - \pi_{t+1} - \rho\end{aligned}$$

- and then

$$1 = E_t \left[\exp \left\{ i_t - \sigma \Delta c_{t+1} - \pi_{t+1} - \rho \right\} \right]$$

$$\Longleftrightarrow c_t = E_t \left[c_{t+1} \right] - \frac{1}{\sigma} \left(i_t - E_t \left[\pi_{t+1} \right] - \rho \right)$$

3.1.3 Linearization of Equation (b)

$$\frac{M_t}{P_t} = C_t^{\sigma/\nu}(1-Q_t)^{-1/\nu} \qquad (b)$$

$$\Longleftrightarrow \exp(m_t - p_t) = \exp\left(\frac{\sigma}{\nu}c_t - \frac{1}{\nu}\log(1-\exp(-i_t))\right)$$

• Taking a first order approximation of each side around the non-stochastic steady state, and using the fact that at this SS

$$\exp(m-p) = \exp\left(\frac{\sigma}{\nu}c - \frac{1}{\nu}\log(1-\exp(-i))\right):$$

$$\exp(m_t - p_t) \approx \exp(m - p) + \exp(m - p)\left((m_t - p_t) - (m - p)\right)$$

and

$$\exp\left(\frac{\sigma}{\nu}c_t - \frac{1}{\nu}\log(1 - \exp(-i_t))\right) =$$

$$\exp(m - p) + \exp(m - p)\left[\frac{\sigma}{\nu}(c_t - c) - \frac{1}{\nu}\frac{-\exp(-i)}{1 - \exp(-i)}(i_t - i)\right]$$

- rearranging terms, we get

$$m_t - p_t = \frac{\sigma}{\nu}c_t - \eta i_t$$

with

$$\eta = \frac{1}{\nu(exp(i) - 1)} \approx \frac{1}{\nu i}$$

3.1.4 Putting Everything Together

- We assume in the following a unit income (consumption) elasticity of money demand: $\sigma = \nu$
- The optimal Hh behavior is then summarized by the three following equations (+ the BC)

$$\begin{cases} w_t - p_t = \sigma c_t + \phi n_t & (1) \quad \text{labor supply} \\ m_t - p_t = c_t - \eta i_t & (2) \quad \text{money demand} \\ c_t = E_t\left[c_{t+1}\right] - \frac{1}{\sigma}\left(i_t - E_t\left[\pi_{t+1}\right] - \rho\right) & (3) \quad \text{consumption/saving} \end{cases}$$

3.2 Firms

- $Y_t = A_t N_t^{1-\alpha}$
- $\Pi_t = P_t Y_t - W_t N_t$
- FOC: $\frac{W_t}{P_t} = (1-\alpha) A_t N_t^{-\alpha}$
- From which we obtain the log-linear labor demand (+ the production function):

$$\begin{cases} w_t - p_t = a_t - \alpha n_t + \log(1-\alpha) & (4) \quad \text{labor demand} \\ y_t = a_t + (1-\alpha) n_t & (5) \quad \text{production function} \end{cases}$$

3.3 Equilibrium – Real Variables

- The real equilibrium is given by the good market equilibrium and the labor market one (+ bonds market clearing that implies $B_t = 0 \ \forall t$).

$$\begin{cases} a_t - \alpha n_t + \log(1-\alpha) = w_t - p_t = \sigma c_t + \phi n_t & \text{labor market} \\ c_t = y_t = a_t + (1-\alpha) n_t & \text{good market} \end{cases}$$

which gives

$$\begin{cases} n_t = \psi_{na} a_t + \theta_n \\ y_t = \psi_{ya} a_t + \theta_y \end{cases}$$

with $\psi_{na} = \frac{1-\sigma}{\sigma(1-\alpha)+\phi+\alpha}$, $\psi_{ya} = \frac{1+\phi}{\sigma(1-\alpha)+\phi+\alpha}$, $\theta_n = \frac{\log(1-\alpha)}{\sigma(1-\alpha)+\phi+\alpha}$, $\theta_y =$

$(1-\alpha)\theta_n$.

- and then

$$\begin{cases} r_t &= i_t - E_t\left[\pi_{t+1}\right] \\ &= \sigma E_t\left[\Delta y_{t+1}\right] + \rho \\ &= \rho + \sigma\psi_{ya} E_t\left[\Delta a_{t+1}\right] \\ \\ \omega_t &= w_t - p_t \\ &= a_t - \alpha n_t + \log(1-\alpha) \\ &= \psi_{\omega a} a_t + \theta_\omega \end{cases}$$

with $\psi_{a\omega} = \frac{\sigma+\phi}{\sigma(1-\alpha)+\phi+\alpha}$, $\theta_\omega = \frac{(\sigma(1-\alpha)+\phi)\log(1-\alpha)}{\sigma(1-\alpha)+\phi+\alpha}$.

Discussion:

- Equilibrium real allocations are independent of monetary policy

- A tech. shock increases y,

- A tech. shock has an ambiguous effect on n (wealth versus substitution effect)

- A tech. shock has an ambiguous effect on r , depending on whether $E_t a_{t+1} \gtrless a_t$

3.4 Equilibrium – Nominal Variables

• We have in equilibrium the "Fisherian" (Irving Fisher) equation : $i_t = E_t\pi_{t+1} + r_t$

• Monetary policy is about

- choosing i (and therefore to π, as r is given from the real side of the economy)

- or choosing m, which determines p, π and i.

3.4.1 A Nominal Interest Rate Policy

- The "Monetary Authorities" control i, that follows an arbitrary exogenous stationary process $\{i_t\}$
- w.l.o.g, $\{i_t\}$ has mean ρ, and $\gamma = 0$, which is consistent with zero-inflation SS.
- A particular case is $i_t = i = \rho$
- With given $\{i_t\}$, the Fisher equation implies

$$E_t\pi_{t+1} = \underbrace{i_t}_{\text{determined by monetary policy}} - \underbrace{r_t}_{\text{determined from the real side of the model}} \tag{6}$$

- We have equation for expected expectation, **but not for actual inflation**

- Let ξ_t be any **non fundamental** shocks (sunspot) s.t. $E_t\xi_{t+1} = 0$,
- Then $p_{t+1} = p_t + i_t - r_t + \xi_{t+1}$
- The price level is **indeterminate**, as it can be affected by any non-fundamental shock.
- Indeterminacy is only nominal
- The money supply that implements the interest rate policy can be recovered from money demand (+ the fact that money demand = money supply in equilibrium):

$$m_t = p_t + y_t - \eta i_t$$

3.4.2 An Inflation-Based Nominal Interest Rule

- Assume $i_t = \rho + \phi_\pi \pi_t, \qquad \phi_\pi \geq 0$
- Using (6),

$$\phi_\pi \pi_t = E_t \pi_{t+1} + \underbrace{\widehat{r}_t}_{r_t - \rho}$$

- The solution of this equation depends on the value of ϕ_π

$\phi_\pi > 1$ **:** In that case, we iterate forward:

$$\pi_t = \sum_{j=0}^{\infty} \phi_\pi^{-(j+1)} E_t \widehat{r}_{t+j}$$

which fully determines π_t.

If for example

$$a_t = \rho_a a_{t-1} + \varepsilon_t^a,$$

then

$$\widehat{r}_t = -\sigma\psi_{ya}(1-\rho_a)a_t$$

and

$$\pi_t = \frac{\sigma\psi_{ya}(1-\rho_a)}{\phi_\pi - \rho_a}a_t.$$

- The larger is ϕ_π, the smaller the volatility of inflation

$\phi_\pi < 1$ **:** In that case,

$$E_t\pi_{t+1} = \phi_\pi\pi_t - \widehat{r}_t.$$

• Again, any **non fundamental** shocks ξ_t s.t. $E_t \xi_{t+1} = 0$ can be added to inflation:

$$\pi_{t+1} = \phi_\pi \pi_t - \widehat{r}_t + \xi_{t+1},$$

and inflation and the price level are indeterminate.

TAYLOR **principle :** *The Monetary Authorities must respond "aggressively" to inflation ($\phi_\pi > 1$) to guarantee determinacy of the price level (and inflation).*

3.5 Monetary Rule

- The Monetary Authorities directly choose a path $\{m_t\}$
- From the money demand equation

$$m_t - p_t = y_t - \eta i_t,$$

we get

$$i_t = \frac{1}{\eta}(m_t - p_t - y_t). \tag{7}$$

- (6) writes

$$i_t = E_t p_{t+1} - p_t + r_t$$

- Combining with (7):

$$\frac{1}{\eta}(m_t - p_t - y_t) = E_t p_{t+1} - p_t + r_t$$

$$\Longleftrightarrow p_t = \frac{\eta}{1+\eta} E_t p_{t+1} + \frac{1}{1+\eta} m_t + u_t$$

where $u_t = (1+\eta)^{-1}(\eta r_t - y_t)$ is independent from m_t.

- Assuming $\eta > 0$, then $\frac{\eta}{1+\eta} < 1$ and we can solve forward to get

$$p_t = \frac{1}{1+\eta} \sum_{j=0}^{\infty} \left(\frac{\eta}{1+\eta} \right)^j E_t m_{t+j} + \widehat{u}_t$$

with $\widehat{u}_t = \sum_{j=0}^{\infty} \left(\frac{\eta}{1+\eta} \right)^j E_t u_{t+j}$.

- The price level is fully determined, and can be written as

$$p_t = m_t + \sum_{j=1}^{\infty} \left(\frac{\eta}{1+\eta}\right)^j E_t \Delta m_{t+j} + \widehat{u}_t$$

- Using the money demand equation, we can obtain the solution

for the nominal interest rate:

$$\begin{aligned} i_t &= \eta^{-1}(y_t - (m_t - p_t) \\ &= \eta^{-1} \textstyle\sum_{j=1}^{\infty} \left(\frac{\eta}{1+\eta}\right)^j E_t \Delta m_{t+j} + \widetilde{u}_t \end{aligned}$$

with $\widetilde{u}_t = \eta^{-1}(\widehat{u}_t + y_t)$.

Example : Assume $\Delta m_t = \rho_m \Delta m_{t-1} + \varepsilon_t^m$ with $0 < \rho_m < 1$

- Then $p_t = m_t + \frac{\eta \rho_m}{1+\eta(1-\rho_m}\Delta m_t$

- Note that

$$\frac{\partial p_t}{\partial \varepsilon_t^m} = 1 + \frac{\eta \rho_m}{1 + \eta(1 - \rho_m} > 1$$

- The price level is predicted to respond more than one to one wrt money shocks $\rightsquigarrow$ not supported by the data, that show price stickiness.

3.6 Optimal Monetary Policy

- Money has no impact on real allocations...
- but $\frac{M}{P}$ enters the utility function.
- Let's solve a Social Planner problem.
- Note that the SP problem is static

$$\begin{array}{ll} \max_{C,M,N} & U\left(C_t, N_t, \frac{M_t}{P_t}\right) \\ \text{s.t.} & C_t = A_t N_t^{1-\alpha} \end{array}$$

- FOC:

$$\begin{cases} -\frac{U_{n,t}}{U_{c,t}} = (1-\alpha) A_t N_t^{-\alpha} & (aa) \\ U_{m,t} = 0 & (bb) \end{cases}$$

Comments :

- (aa) holds in competitive equilibrium

- (bb) means that, given that producing real balances is free, the SP should satiate the agents ("choose $P = \infty$")

- the competitive equivalent of (bb) is

$$\frac{U_{m,t}}{U_{c,t}} = 1 - Q_t = 1 - \exp(i_t)$$

- To implement a social optimum, monetary policy should set $i_t = 0\ \forall t$, so that $\pi_t = -\rho$.

FRIEDMAN **rule :** *Disinflation is optimal, at a rate equal to minus the real interest rate. This guarantees that the nominal interest rate (which is the opportunity cost of holding money) is zero.*

Remark : A rule $i_t = 0$ leads to indeterminate inflation and prices.

• The rule $i_t = \phi(r_{t-1} + \pi_t)$ with $\phi > 1$ allows for determinacy, and $i_t = r_t + E_t \pi_{t+1}$ implies

$$E_t i_{t+1} = \phi E_t [r_t + \pi_{t+1}] = \phi i_t \Rightarrow i_t = 0 \quad \text{(solving forward)}$$

and inflation is determined as $\pi_t = -r_{t-1}$

4 The Simple New-Keynesian Model

4.1 Household

- Objective :

$$E_0 \sum_{t=0}^{\infty} \beta^t V\left(C_t, N_t, \frac{M_t}{P_t}\right)$$

- We assume in the following: $V = \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1-\phi}}{1-\phi} + \chi \frac{\left(\frac{M_t}{P_t}\right)^{1-\sigma}}{1-\sigma}$
- We also assume $\chi \to 0$ so that $V\left(C_t, N_t, \frac{M_t}{P_t}\right) \approx U\left(C_t, N_t\right)$ but there exist a well-defined money demand function.
- Budget Constraint :

$$P_t C_t + M_t + Q_t B_t \leq B_{t-1} + W_t N_t + M_{t-1} + T_t$$

- $\mathcal{A}_t = B_{t-1} + M_{t-1}$, $\lim_{T\to\infty} E_t \mathcal{A}_T \geq 0$

4.1.1 Optimal C, N and M

$$\begin{cases} w_t - p_t = \sigma c_t + \phi n_t & (1) \quad \text{labor supply} \\ m_t - p_t = c_t - \eta i_t & (2) \quad \text{money demand} \\ c_t = E_t\left[c_{t+1}\right] - \frac{1}{\sigma}\left(i_t - E_t\left[\pi_{t+1}\right] - \rho\right) & (3) \quad \text{consumption/saving} \end{cases}$$

4.1.2 Optimal Composition of C

- We assume that C is a basket of a continuum of consumption goods which are imperfect substitutes:

$$C_t = \left(\int_0^1 C_t(i)^{1-\frac{1}{\varepsilon}} di\right)^{\frac{\varepsilon}{\varepsilon-1}}$$

with prices $P_t(i)$ such that $P_t C_t = \int_0^1 P_t(i) C_t(i) di$

- Optimal composition of the basket is obtained by following the following program:

$$\begin{array}{ll} \min & \int_0^1 P_t(i) C_t(i) di \\ \text{s.t.} & \left(\int_0^1 C_t(i)^{1-\frac{1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}} \geq C_t \end{array}$$

- The solution of this problem is

$$C_t(i) = \left(\frac{P_t(i)}{P_t} \right)^{-\varepsilon} C_t \qquad (e)$$

with

$$P_t = \left(\int_0^1 P_t(i)^{1-\varepsilon} \right)^{\frac{1}{1-\varepsilon}}$$

Details of the algebra: (I am dropping the time subscript)

$$\min \int_0^1 P_i C_i di \qquad s.t. \qquad \left(\int_0^1 (C_i)^{\frac{\varepsilon-1}{\varepsilon}} di\right)^{\frac{\varepsilon}{\varepsilon-1}} \geq C \qquad (\lambda)$$

The FOC of this program with respect to C_j gives

$$P_j = \lambda C_j^{-\frac{1}{\varepsilon}} \left(\int_0^1 (C_i)^{\frac{\varepsilon-1}{\varepsilon}} di\right)^{\frac{\varepsilon}{\varepsilon-1}-1}$$

Taking the ratio of the FOC with respect to i with the FOC with respect to j gives

$$\frac{P_i}{P_j} = \left(\frac{C_i}{C_j}\right)^{-\frac{1}{\varepsilon}}$$

which is equivalent to

$$C_i = C_j \left(\frac{P_i}{P_j}\right)^{-\varepsilon}$$

Use this expression of C_i in the constraint (that is binding)

$$\left(\int_0^1 (C_i)^{\frac{\varepsilon-1}{\varepsilon}} di\right)^{\frac{\varepsilon}{\varepsilon-1}} = C,$$

to obtain

$$C = \left[\int_0^1 \left(C_j \left(\frac{P_i}{P_j}\right)^{-\varepsilon}\right)^{\frac{\varepsilon-1}{\varepsilon}} di\right]^{\frac{\varepsilon}{\varepsilon-1}}$$

which is equivalent to

$$C = C_j P_j^{\varepsilon} \left[\left(\int_0^1 P_i^{1-\varepsilon} di\right)^{\frac{1}{1-\varepsilon}}\right]^{-\varepsilon}$$

One recognize the expression of the price level P, such that one gets

$$C_j = \left(\frac{P_j}{P}\right)^{-\varepsilon} C$$

4.2 Firms

- Each firm i is a monopoly, that faces demand function (e), taking C_t and P_t as given (monopolistic competition)

$$Y(i)_t = A_t N_t(i)^{1-\alpha}$$

4.2.1 Sticky Prices

- The modeling is the one of Calvo (1983)
- Each period, a firm has a probability $1-\theta$ of being allowed to reset its price
- In the aggregate, a fraction $1-\theta$ of firms resets price, a fraction θ does not.

- Duration of a price:

length in period	probability
1	$(1-\theta)$
2	$\theta(1-\theta)$
3	$\theta^2(1-\theta)$
$\dots$	$\dots$

$$
\begin{aligned}
\text{Average duration} &= 1\times(1-\theta)+2\times\theta(1-\theta)+3\times\theta^2(1-\theta)+4\times\theta \\
&= (1+2\theta+3\theta^2+4\theta^3+\cdots)\times(1-\theta) \\
&= (1+\theta+\theta^2+\theta^3+\cdots \\
&\quad +\theta+\theta^2+\theta^3+\cdots \\
&\quad +\theta^2+\theta^3+\cdots \\
&\quad +\theta^3+\cdots \\
&\quad +\cdots)\times(1-\theta) \\
&= \left(\tfrac{1}{1-\theta}+\tfrac{\theta}{1-\theta}+\tfrac{\theta^2}{1-\theta}+\cdots\right)\times(1-\theta) \\
&= 1+\theta+\theta^2+\theta^3+\cdots \\
&= \tfrac{1}{1-\theta}
\end{aligned}
$$

- θ is therefore an index of price stickiness.

4.2.2 Optimal price setting

- A firm reoptimizing in period t chooses $P_t^\star$ in order to

$$\max_{P_t^\star} \sum_{j=0}^{\infty} \theta^j E_t \left[Q_{t,t+j} \left(P_t^\star Y_{t+j|t} - \Psi_{t+j}(Y_{t+j|t}) \right) \right]$$

$$\text{s.t.} \qquad Y_{t+j|t} = \left(\frac{P_t^\star}{P_{t+j}} \right)^{-\varepsilon} C_{t+j}$$

$$Q_{t,t+j} = \beta^j \left(\frac{C_{t+j}}{C_t} \right)^{-\sigma} \left(\frac{P_t}{P_{t+j}} \right)$$

where $Y_{t+j|t}$ is the demand addressed in $t+j$ to a firm that set its price in t, and $\Psi_{t+j}(\cdot)$ is the cost function is $t+j$

- Consider a period j profit maximization problem:

$$\max_{P_t^\star} \; P_t^\star Y_{t+j|t} - \Psi_{t+j}(Y_{t+j|t})$$

$$\text{s.t.} \qquad Y_{t+j|t} = \left(\frac{P_t^\star}{P_{t+j}}\right)^{-\varepsilon} C_{t+j}$$

- The FOC is

$$Y_{t+j|t} + P_t^\star \times (-\varepsilon)(P_t^\star)^{-\varepsilon-1}\left(\frac{1}{P_{t+j}}\right)^{-\varepsilon} C_{t+j} - \underbrace{\Psi'}_{\psi} \times (-\varepsilon)(P_t^\star)^{-\varepsilon-1}\left(\frac{1}{P_{t+j}}\right)^{-\varepsilon}$$

$$\Longleftrightarrow Y_{t+j|t}\left(P_t^\star - \underbrace{\frac{\varepsilon}{1-\varepsilon}}_{\mathcal{M}} \psi_{t+j|t}\right) = 0$$

- The FOC of the intertemporal problem is therefore:

$$\sum_{j=0}^{\infty} \theta^j E_t \left[Q_{t,t+j} Y_{t+j|t} \left(P_t^\star - \mathcal{M} \psi_{t+j|t} \right) \right] = 0$$

Remark 1 : If $\theta = 0$, then $P_t^\star = \mu \psi_t$.

Remark 2 : The choice of $P_t^\star$ is purely forward-looking. All firms resetting price will choose the same $P_t^\star$

- Rewrite the FOC in real terms, divide by P_{t-1}, and use $\Pi_{t+j,t} =$

$\frac{P_{t+j}}{P_t}$:

$$\sum_{j=0}^{\infty} \theta^j E_t \left[Q_{t,t+j} Y_{t+j|t} \left(\frac{P_t^\star}{P_{t-1}} - \mathcal{M} \underbrace{MC_{t+j,t}}_{\text{real marginal cost}} \Pi_{t-1,t+j} \right) \right] = 0 \qquad (d)$$

- Consider again a zero inflation SS. At the SS, we have

$$\frac{P_t^\star}{P_{t-1}} = 1 \quad , \quad \Pi_{t-1,t+j} = 1 \quad , \quad P_t^\star = P_{t+j} \quad \forall \;\; t, j$$

and therefore

$$Y_{t+j,t} = Y \;\; , \;\; MC_{t+j,t} = MC \;\; , \;\; Q_{t,t+j} = \beta^j \;\; , \;\; P^\star = \mathcal{M}\psi = \mathcal{M} \times MC \times P$$

- Take a first order expansion of (d) around the zero inflation SS:

$$p_t^\star - p_{t-1} = (1-\beta\theta)\sum_{j=0}^{\infty}(\beta\theta)^j E_t\left[\widehat{mc}_{t+j|t} + p_{t+j} - p_{t-1}\right]$$

with $\widehat{mc}_{t+j|t} = mc_{t+j|t} - mc$, $mc = -\mu$, $\mu = \log\mathcal{M}$.

- This equation can also be written

$$p_t^\star = \mu + (1-\beta\theta)\sum_{j=0}^{\infty}(\beta\theta)^j E_t\left[mc_{t+j|t} + p_{t+j}\right]$$

- The firm chooses a desired markup over a weighted average of current and expected nominal mc.

4.2.3 Aggregate Price Dynamics

- Let $s(t) \subset [0,1]$ be the set of firms that do not reset their price in period t.

- As seen previously, $P_t^\star$ is the same for all resetting firms, so that

$$P_t = \left[\int_{s(t)} P_{t-1}^{1-\varepsilon}(i)di + (1-\theta)\left(P_t^\star\right)^{1-\varepsilon} \right]^{\frac{1}{1-\varepsilon}}$$

$$\Longleftrightarrow \Pi_t^{1-\varepsilon} = \left[\theta + (1-\theta)\left(\frac{P_t^\star}{P_{t-1}}\right)^{1-\varepsilon} \right]$$

- log-linearizing around the non stochastic zero inflation SS (skipping some algebra):

$$\pi_t = (1-\theta)\left(p_t^\star - p_{t-1}\right)$$

4.3 Equilibrium

The key difference with the "classical" model is that money matters for real equilibrium allocations

4.3.1 Good Market

$$Y_t(i) = C_t(i) \quad \forall \quad i \quad \forall \quad t$$

which gives by aggregation

$$Y_t = C_t$$

and replacing in the Hh Euler equation:

$$y_t = E_t y_{t+1} - \frac{1}{\sigma}\left(i_t - E_t \pi_{t+1} - \rho\right)$$

4.3.2 Labor Market

$$N(t) = \int_0^1 N_t(i)di = \int_0^1 \left(\frac{Y_t(i)}{A_t}\right)^{\frac{1}{1-\alpha}} di = \left(\frac{Y_t}{A_t}\right)^{\frac{1}{1-\alpha}} \int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{\frac{-\varepsilon}{1-\alpha}} di$$

which gives in logs

$$(1-\alpha)n_t = y_t - a_t + d_t$$

where

$$d_t = (1-\alpha)\log\left[\int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{\frac{-\varepsilon}{1-\alpha}} di\right]$$

- One can show (with some algebra) that around a zero inflation SS, $d_t \approx 0 + \mathcal{O}(d_t^2)$ so that, to a first order,

$$(1-\alpha)n_t = y_t - a_t$$

4.3.3 The New-Keynesian Philips Curve

- Let mc_t be the (log) average marginal cost and mpn_t be the log (average) marginal product of labor:

$$\begin{aligned} mc_t &= (w_t - p_t) - mpn_t \\ &= (w_t - p_t) - (a_t - \alpha n_t + \log(1-\alpha)) \\ &= (w_t - p_t) - \tfrac{1}{1-\alpha}(a_t - \alpha y_t) - \log(1-\alpha) \end{aligned}$$

- For a firm that does not reoptimize between t and $t+j$:

$$\begin{aligned} mc_{t+j|t} &= (w_{t+j} - p_{t+j}) - \tfrac{1}{1-\alpha}(a_{t+j} - \alpha \mathbf{y_{t+j|t}}) - \log(1-\alpha) \\ &= mc_{t+j} + \tfrac{\alpha}{1-\alpha}(y_{t+j|t} - y_{t+j}) \\ &= mc_{t+j} + \tfrac{\alpha\varepsilon}{1-\alpha}(p_t^{\star} - p_{t+j}) \end{aligned}$$

(using firm demand)

- We have shown that the pricing decision was *forward-looking*:

$$p_t^\star - p_{t-1} = (1-\beta\theta)\sum_{j=0}^{\infty}(\beta\theta)^j E_t[\underbrace{\widehat{mc}_{t+j|t}}_{\widehat{mc}_{t+j}+\frac{\alpha\varepsilon}{1-\alpha}(p_t^\star - p_{t+j})} + p_{t+j} - p_{t-1}]$$

- Rearranging terms, we obtain

$$\pi_t = \beta E_t \pi_{t+1} + \lambda \widehat{mc}_t$$

with $\lambda = \frac{(1-\theta)(1-\beta\theta)}{\theta}\Theta$ and $\Theta = \frac{1-\alpha}{1-\alpha+\alpha\varepsilon}$

- note that

$$\lambda = \lambda(\underset{-}{\theta}, \underset{-}{\alpha}, \underset{-}{\varepsilon})$$

- Solving forward:

$$\pi_t = \lambda \sum_{j=0}^{\infty} \beta E_t \widehat{mc}_{t+j}$$

where $\widehat{mc}_{t+j}$ is the average mc and is also $-\mu_t$ (minus the average markup).

- When average markups are expected to be below SS, mc are expected to be high and inflation is high.
- To obtain a Philips Curve-like equation, let's substitute mc for output:

$$mc_t = (w_t - p_t) - mpn_t = \underbrace{(\sigma y_t + \phi n_t)}_{\text{labour demand}} - \underbrace{(y_t - n_t + \log(1-\alpha))}_{\text{labor supply}}$$

$$mc_t = \left(\sigma + \frac{\phi+\alpha}{1-\alpha}\right) y_t - \frac{1+\phi}{1-\alpha} a_t - \log(1-\alpha)$$

- Under flex-price, $mc_t = -\mu$, so that we can recover from the above equation the flex-price level of ouptut, denoted y^n (the natural level of output):

$$y_t^n = \psi_{ya}^n a_t + \theta_y^n$$

and

$$\widehat{mc}_t = mc_t - (-\mu) = \sigma + \left(\frac{\phi+\alpha}{1-\alpha}\right) \underbrace{(y_t - y_t^n)}_{\text{output gap } \widehat{y}_t}$$

Therefore

$$\pi_t = \beta E_t \pi_{t+1} + \kappa \widehat{y}_t$$

4.3.4 The Dynamic IS Equation

- Take the Hh Euler equation and introduce the output gap.

$$y_t = E_t y_{t+1} - \frac{1}{\sigma}(i_t - E_t \pi_{t+1} - \rho)$$

- Define the natural interest rate as the flex-price one:

$$r_t^n = \rho + \sigma E_t \Delta y_{t+1}^n = \rho + \sigma \psi_{ya}^n E_t \Delta a_{t+1}$$

- We obtain

$$\widehat{y}_t = E_t \widehat{y}_{t+1} - \frac{1}{\sigma}(i_t - E_t \pi_{t+1} - r_t^n)$$

4.3.5 Equilibrium Summary

- One can see the model as having a recursive structure:

  - natural real interest rate: $r_t^n = \rho + \sigma E_t \Delta y_{t+1}^n = \rho + \sigma \psi_{ya}^n E_t \Delta a_{t+1}$

  - The NKPC determines π_t for a given path of $\widehat{y}_t$: $\pi_t = \beta E_t \pi_{t+1} + \kappa \widehat{y}_t$

  - The DIS determines $\widehat{y}_t$ for a natural and actual real interest rate: $\widehat{y}_t = E_t \widehat{y}_{t+1} - \frac{1}{\sigma}(i_t - E_t \pi_{t+1} - r_t^n)$

  - A monetary policy equation is still needed, which will be non neutral.

4.4 Equilibrium under an Interest Rate Rule

4.4.1 Computing the Equilibrium

- Assume a TAYLOR rule

$$i_t = \underbrace{\rho}_{\text{consistent with zero SS inflation}} + \phi_\pi \pi_t + \phi_y \widetilde{y}_t + v_t$$

with $\phi_\pi > 0$ and $\phi_y > 0$.

- The full model is given by:

$$\begin{cases} r_t^n &= \rho + \sigma\psi_{ya}^n E_t \Delta a_{t+1} \\ \widetilde{y}_t &= E_t \widetilde{y}_{t+1} - \frac{1}{\sigma}(i_t - E_t \pi_{t+1} - r_t^n) \\ \pi_t &= \beta E_t \pi_{t+1} + \kappa \widetilde{y}_t \\ a_t &= \rho_a a_{t-1} + \varepsilon_t^a \\ v_t &= \rho_v v_{t-1} + \varepsilon_t^v \end{cases}$$

or equivalently

$$\underbrace{\begin{pmatrix} \widetilde{y}_t \\ \pi_t \end{pmatrix}}_{X_t} = \underbrace{\Omega}_{\frac{1}{\sigma+\phi_y+\kappa\phi_\pi}} \underbrace{\begin{bmatrix} \sigma & 1-\beta\phi_\pi \\ \sigma\kappa & \kappa+\beta(\sigma+\phi_t) \end{bmatrix}}_{A} \times \begin{pmatrix} E_t \widetilde{y}_{t+1} \\ E_t \pi_{t+1} \end{pmatrix} + \Omega \underbrace{\begin{pmatrix} 1 \\ \kappa \end{pmatrix}}_{B} \underbrace{(\widehat{r}_t - v_t)}_{z_t}$$

- This equation can be solved forward to obtain a unique solution if and only if eig(A)<1, which is guaranteed if $\kappa(\phi_p i - 1) + (1 - \beta)\phi_y > 0$

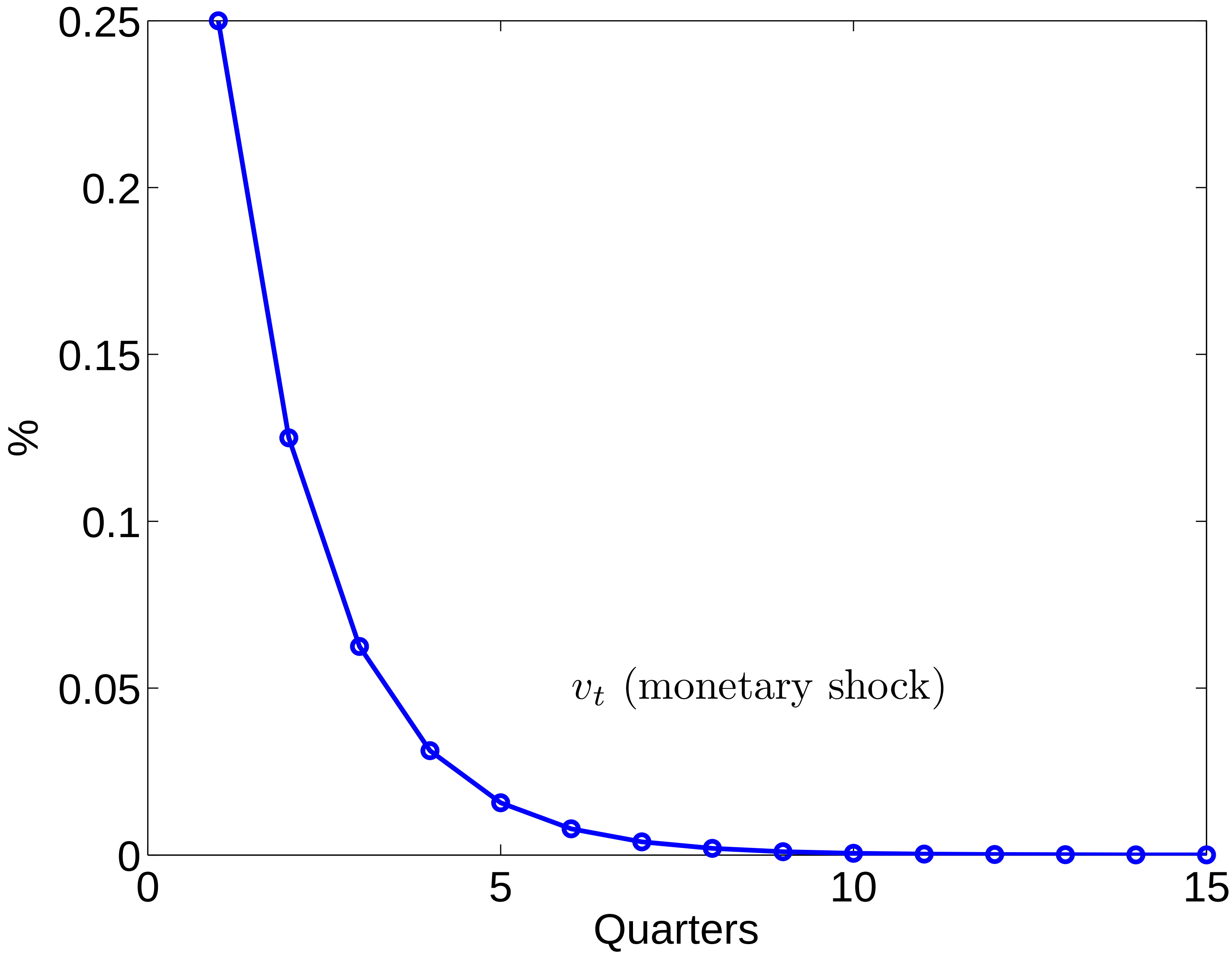
- and the solution is

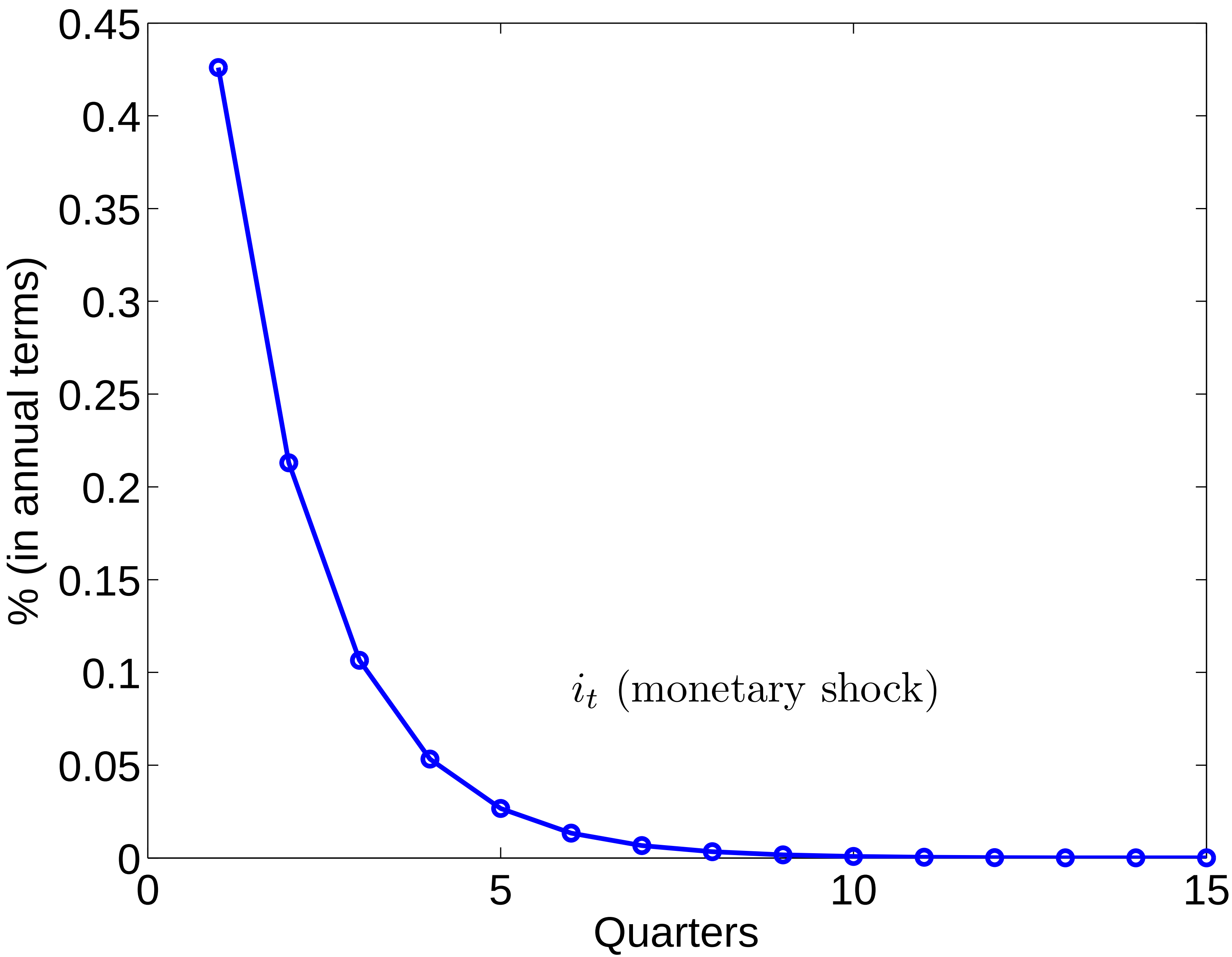
$$X_t = \sum_{j=0}^{\infty} A^j E_t z_{t+j}$$

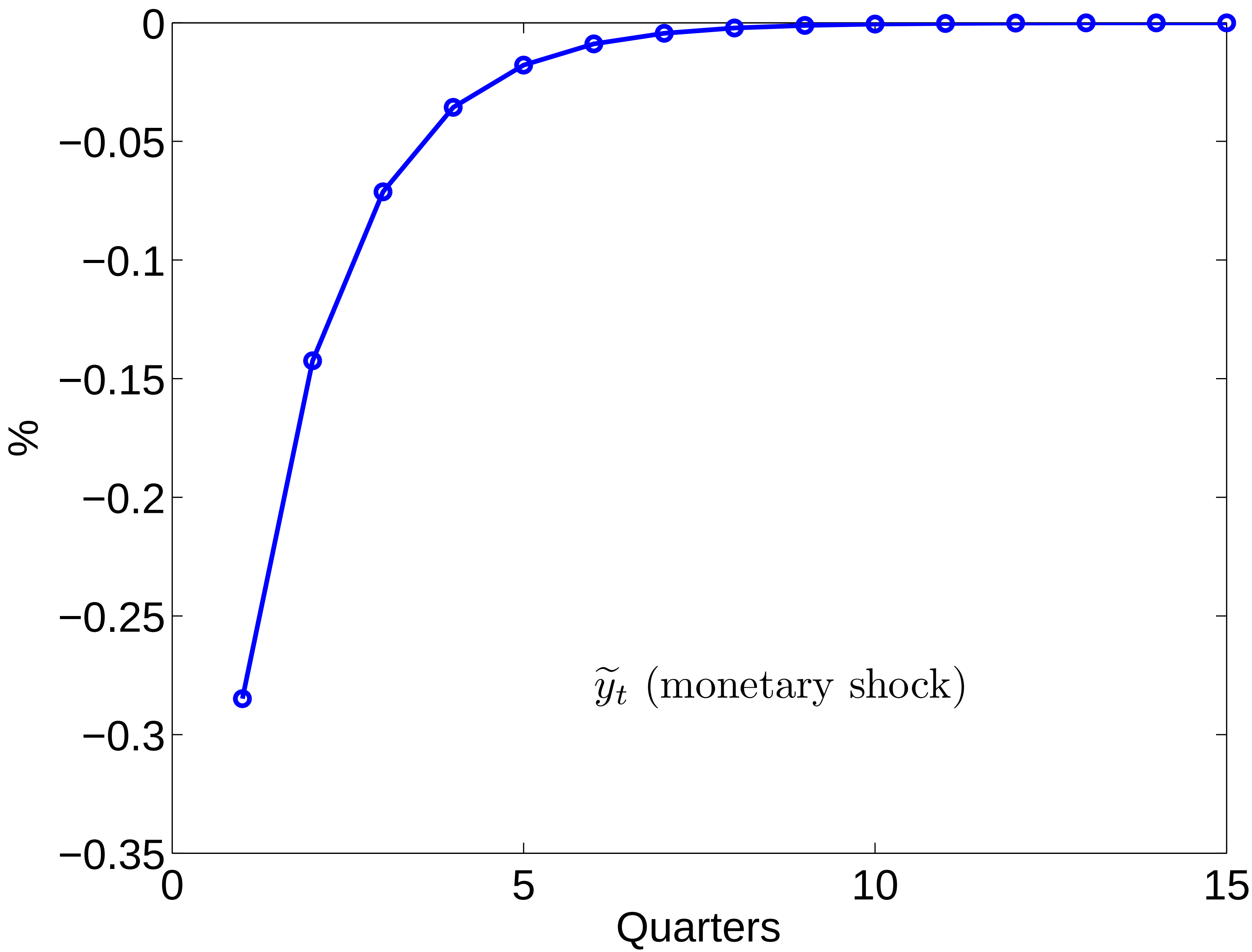
4.4.2 Calibration

β	.99	σ	1
ϕ	1	α	1/3
ε	6	η	4
θ	2/3	ϕ_π	1.5
ϕ_y	.5/4	ρ_a	.9
ρ_v	.5		

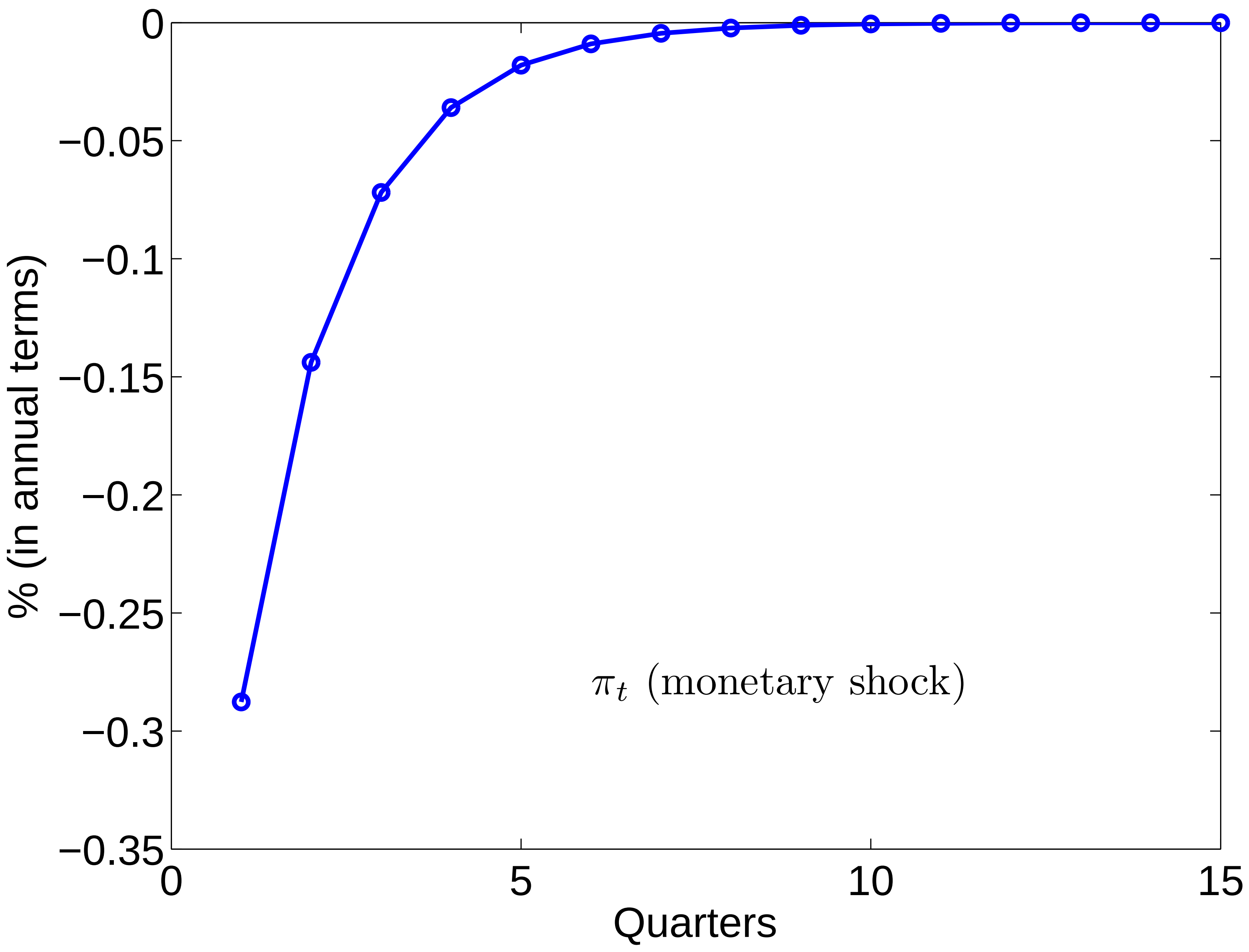
4.4.3 Monetary Shock

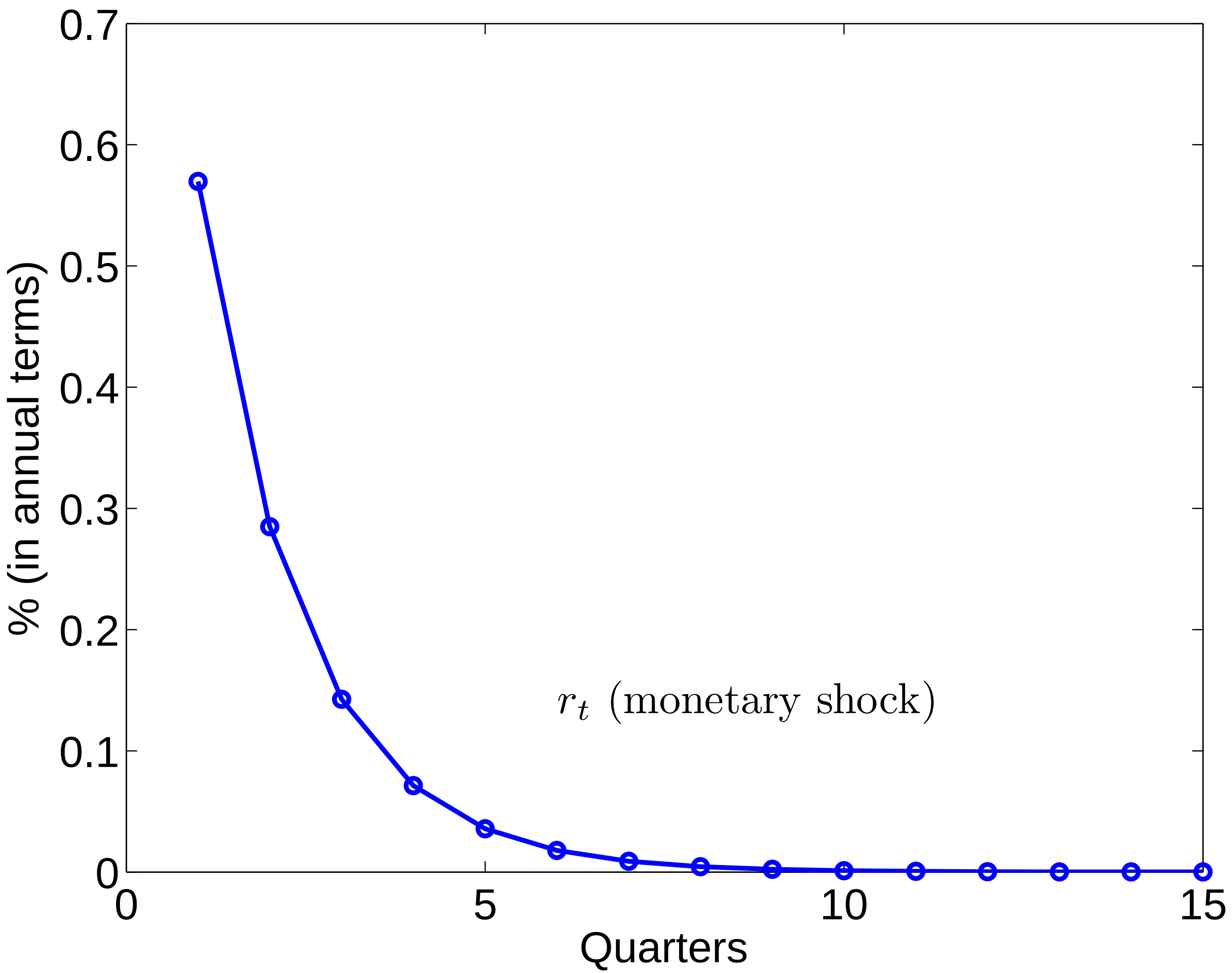



0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
% (in annual terms)
i_t (monetary shock)
0
5
10
15
Quarters



0
−0.05
−0.1
−0.15
−0.2
−0.25
−0.3
−0.35
%
$\widetilde{y}_t$ (monetary shock)
0
5
10
15
Quarters


0
−0.05
−0.1
−0.15
−0.2
−0.25
−0.3
−0.35
% (in annual terms)
0
5
10
15
Quarters
π_t (monetary shock)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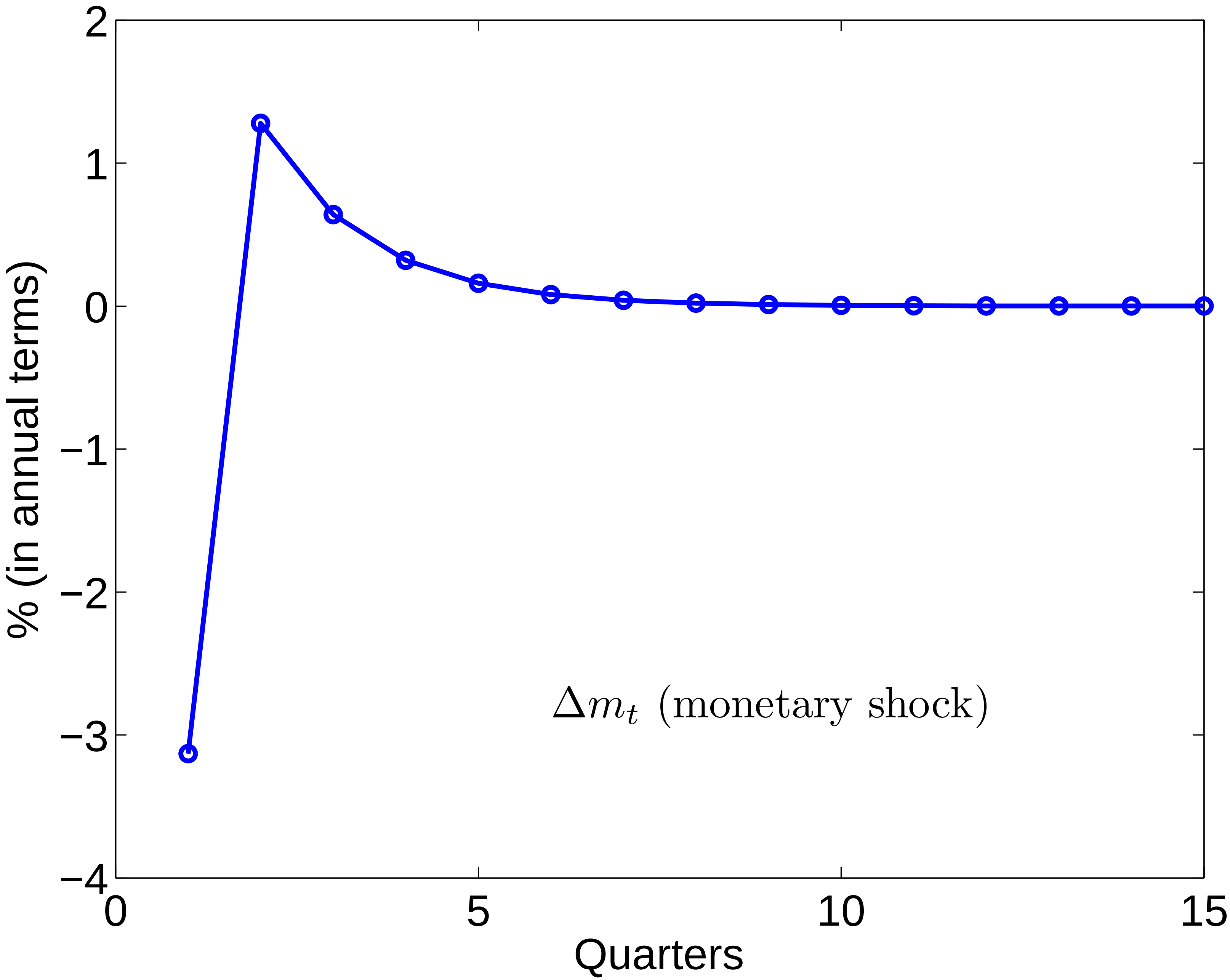
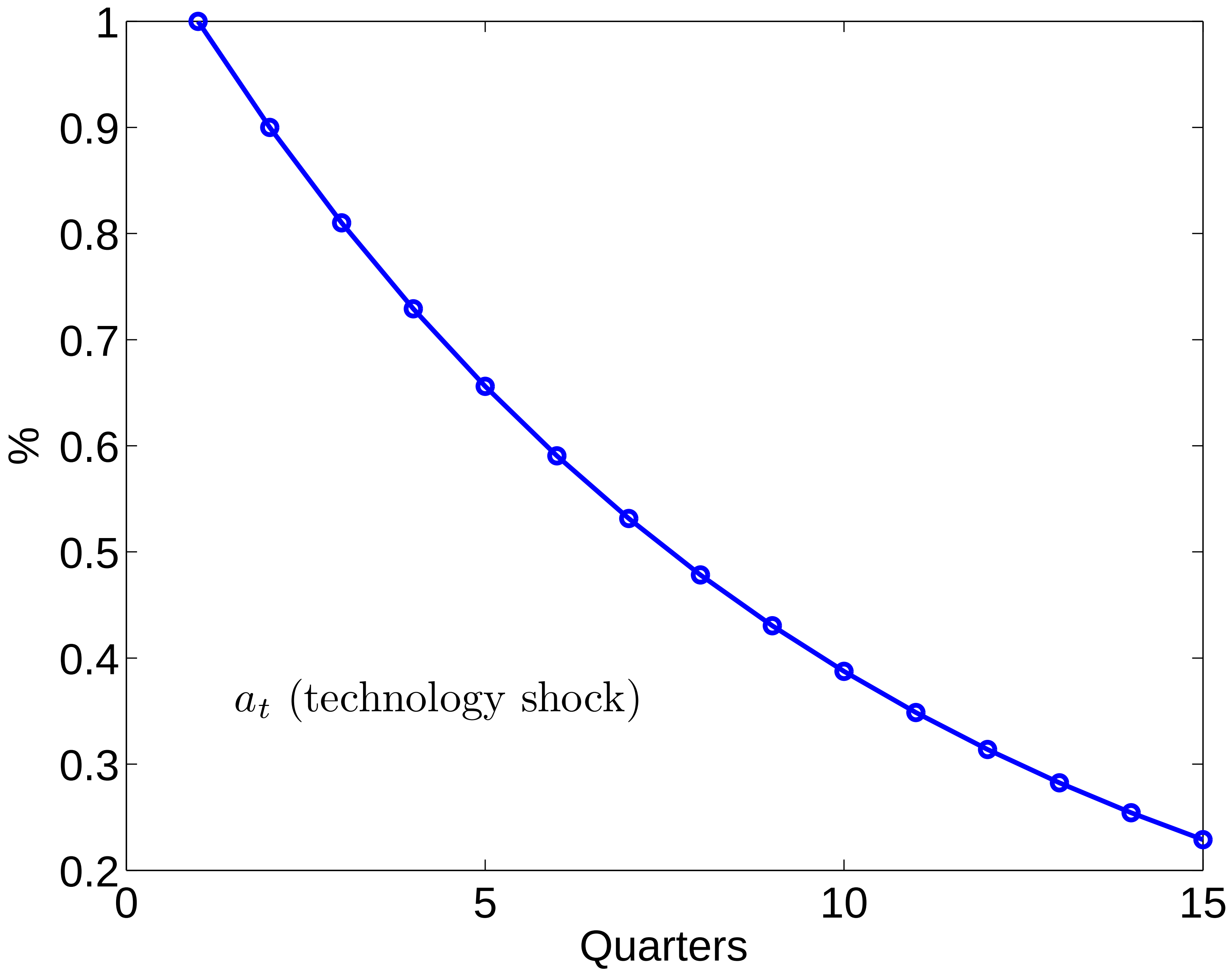
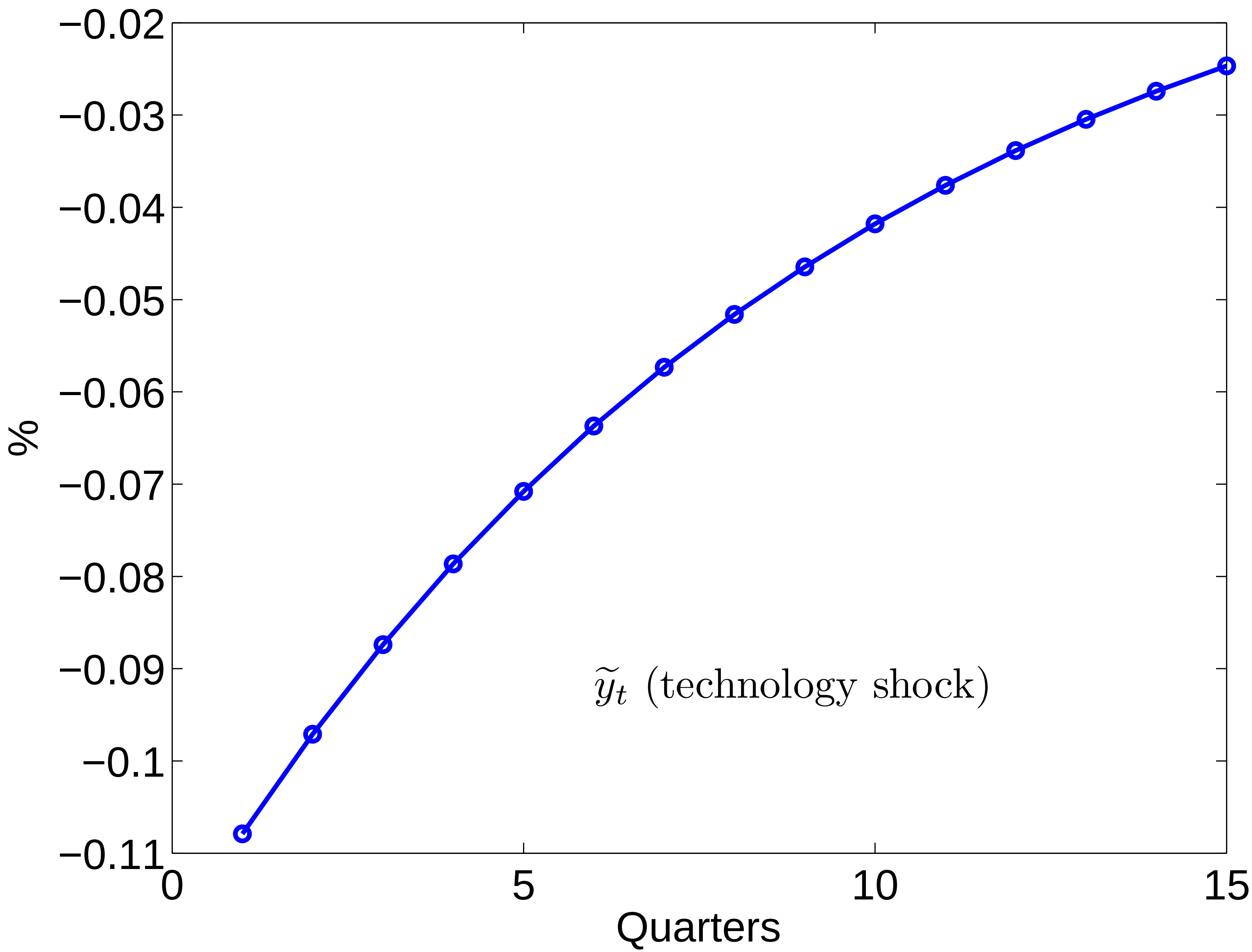
0.6
0.5
0.4
0.3
0.2
0.1
0
0
5
10
15
Quarters
% (in annual terms)
r_t (monetary shock)

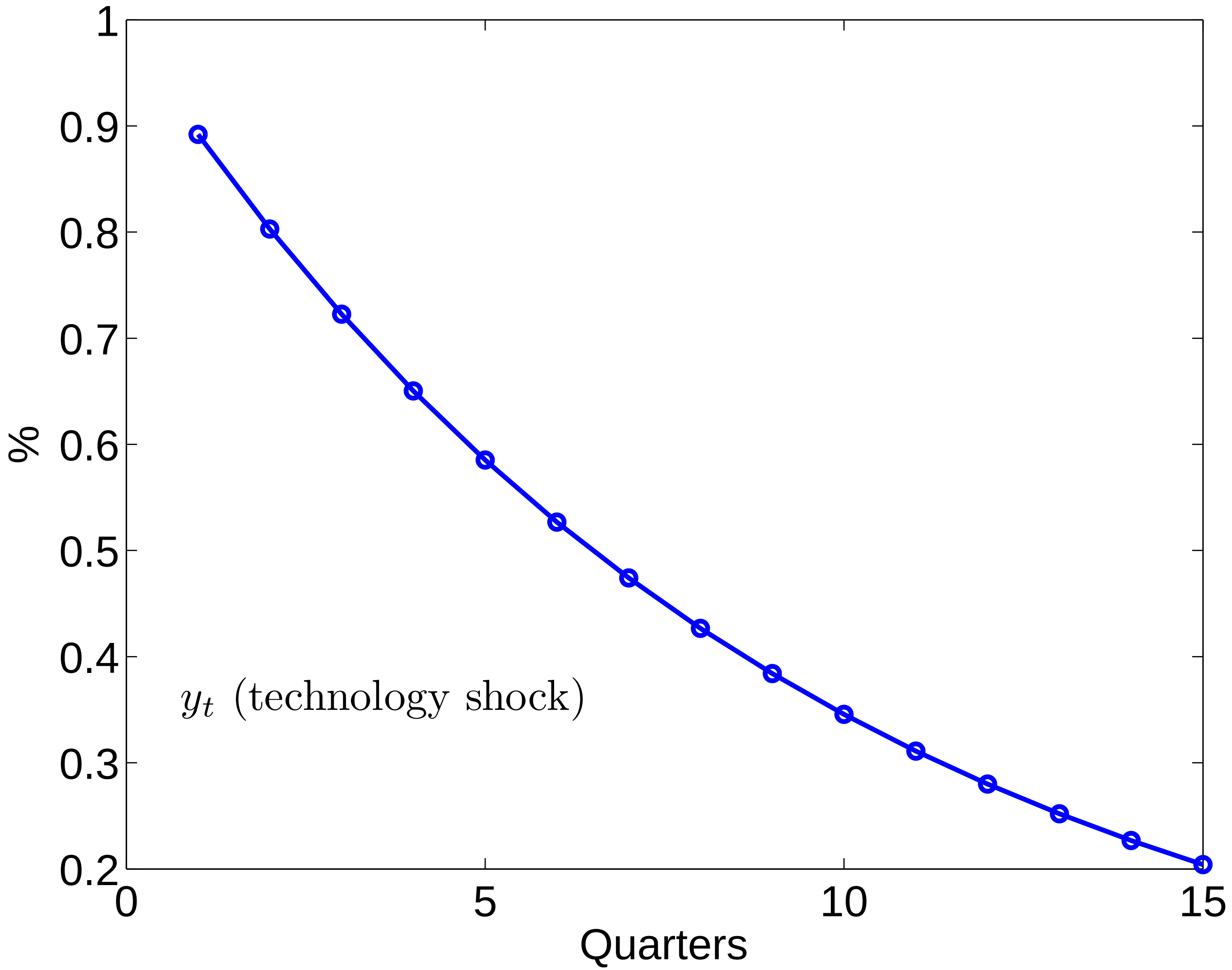

2
1
0
−1
−2
−3
−4
0
5
10
15
Quarters
% (in annual terms)
Δm_t (monetary shock)

4.4.4 Technology Shock





−0.02
−0.03
−0.04
−0.05
−0.06
−0.07
−0.08
−0.09
−0.1
−0.11
%
0
5
10
15
Quarters
$\widetilde{y}_t$ (technology shock)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0.8
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0.4
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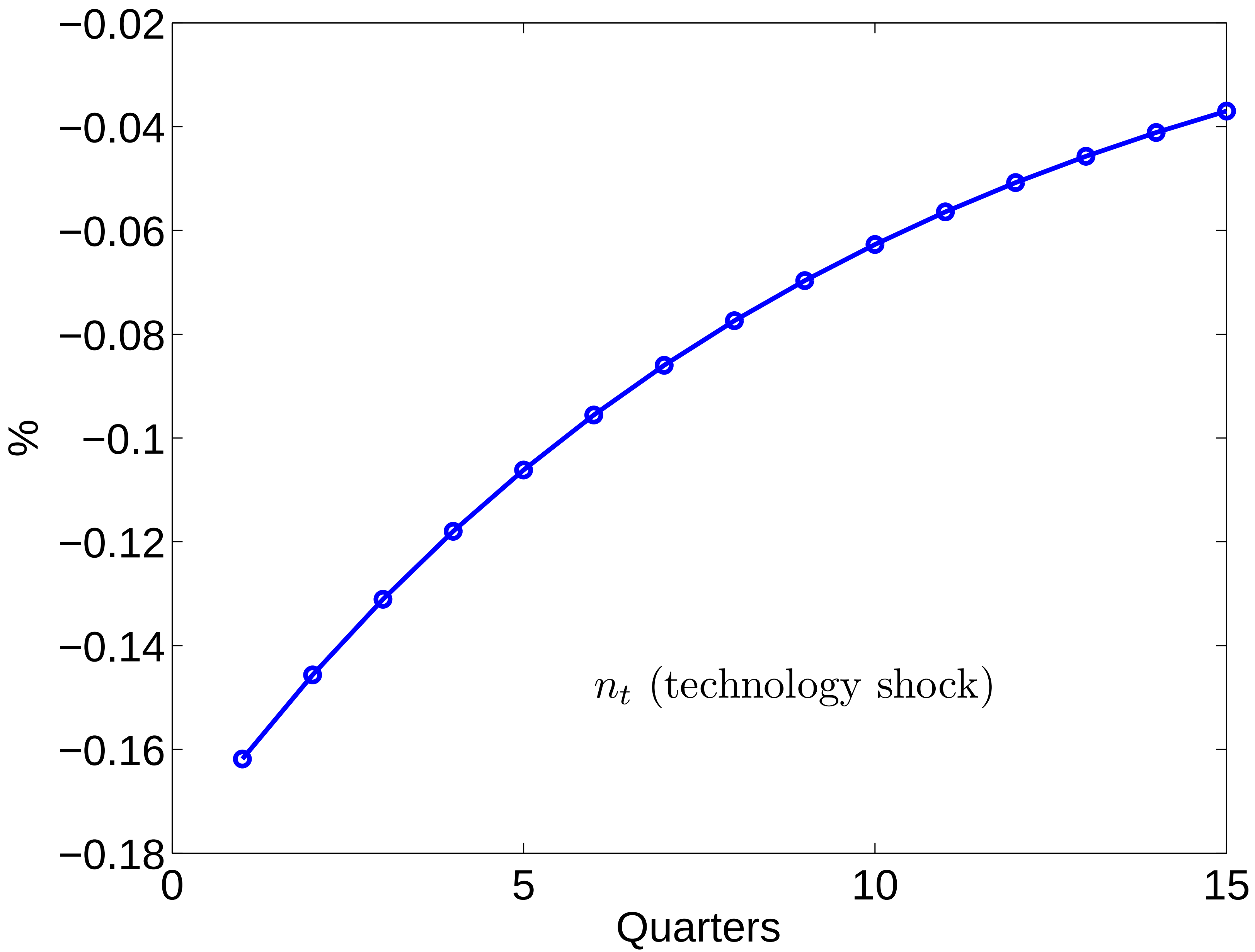
0.2
%
0
5
10
15
Quarters
y_t (technology shock)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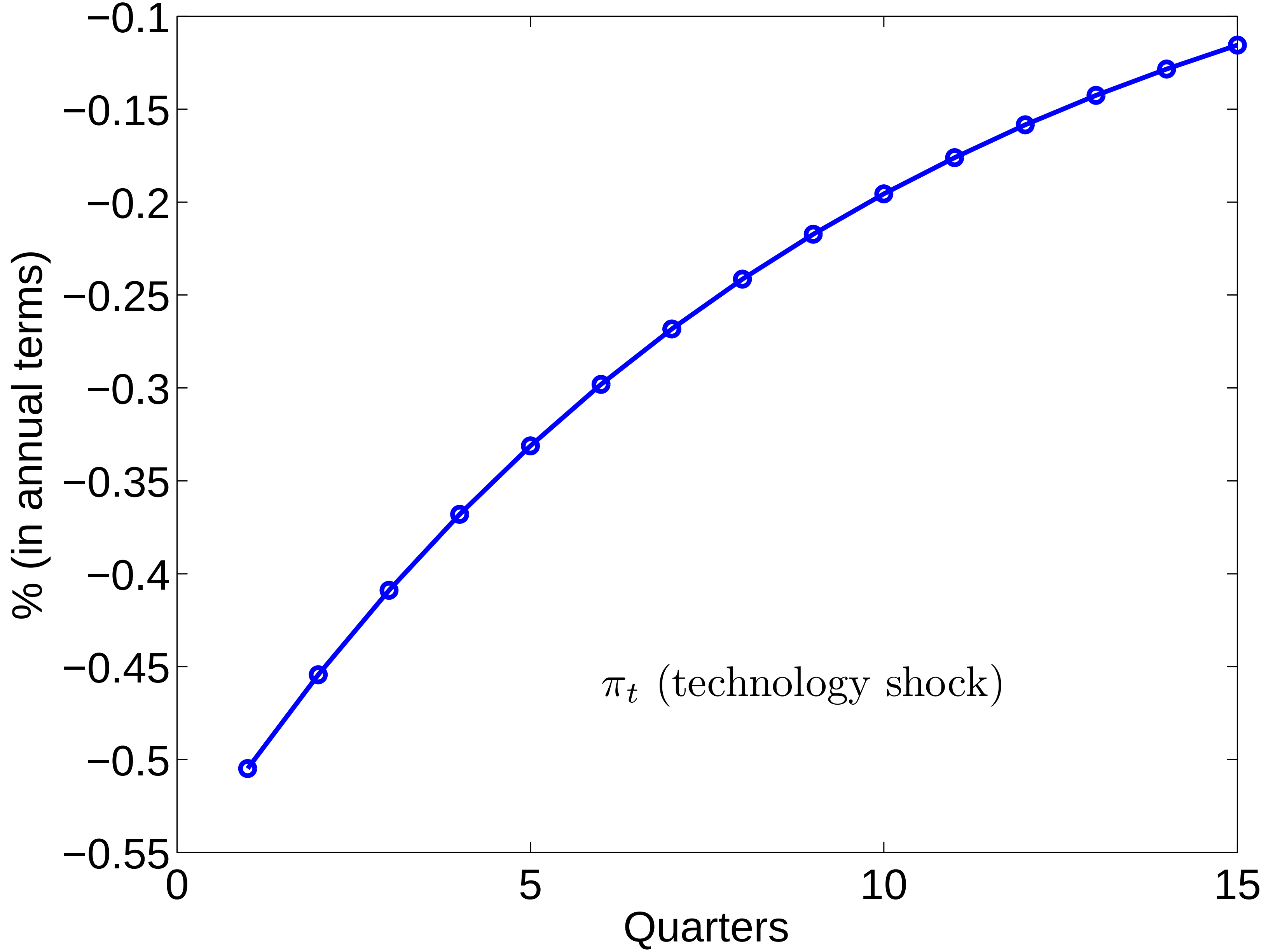
%
0
5
10
15
Quarters
n_t (technology shock)



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−0.2
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−0.3
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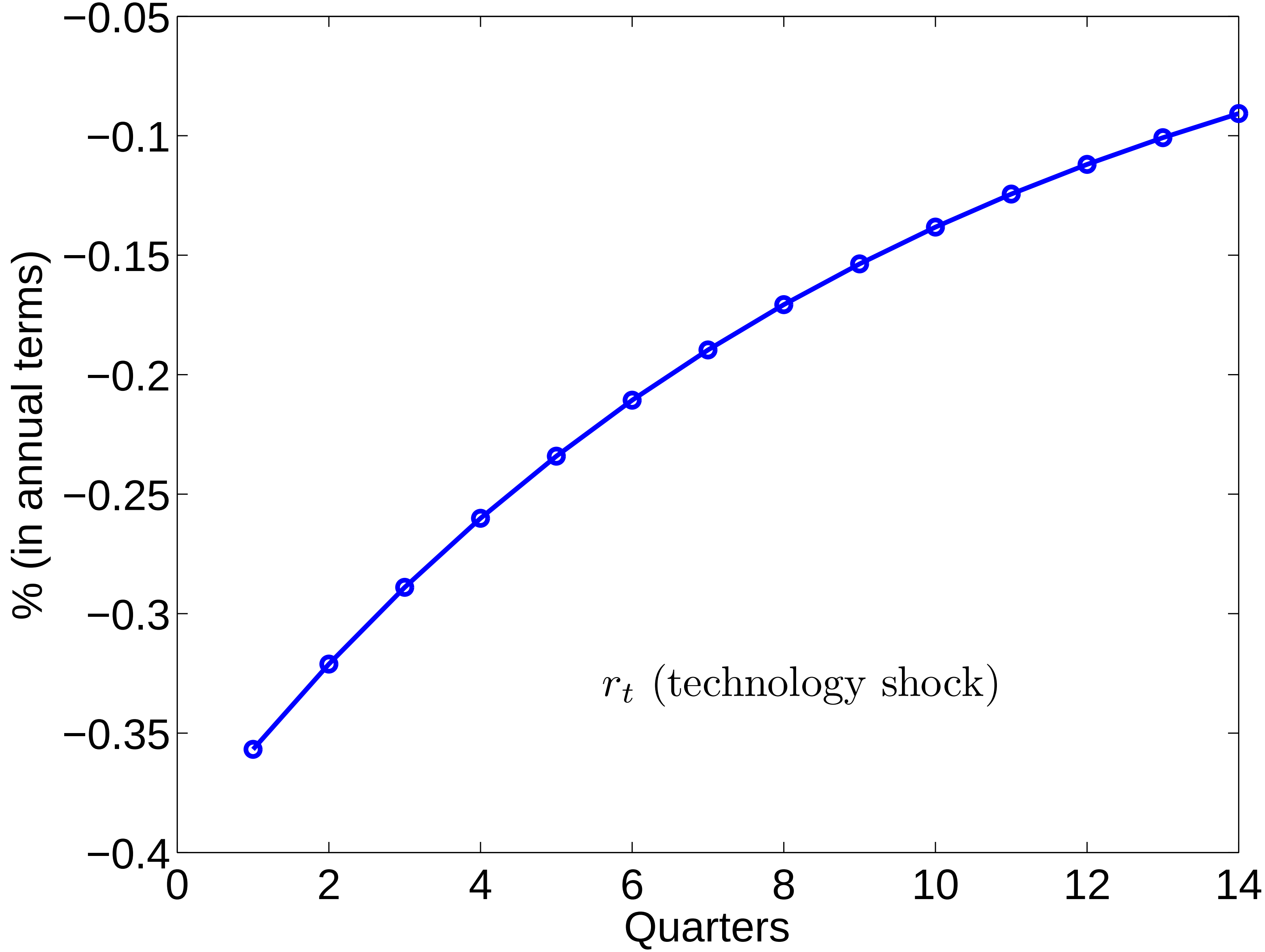
−0.4
−0.45
−0.5
−0.55
% (in annual terms)
0
5
10
15
Quarters
π_t (technology shock)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−0.25
−0.3
−0.35
−0.4
% (in annual terms)
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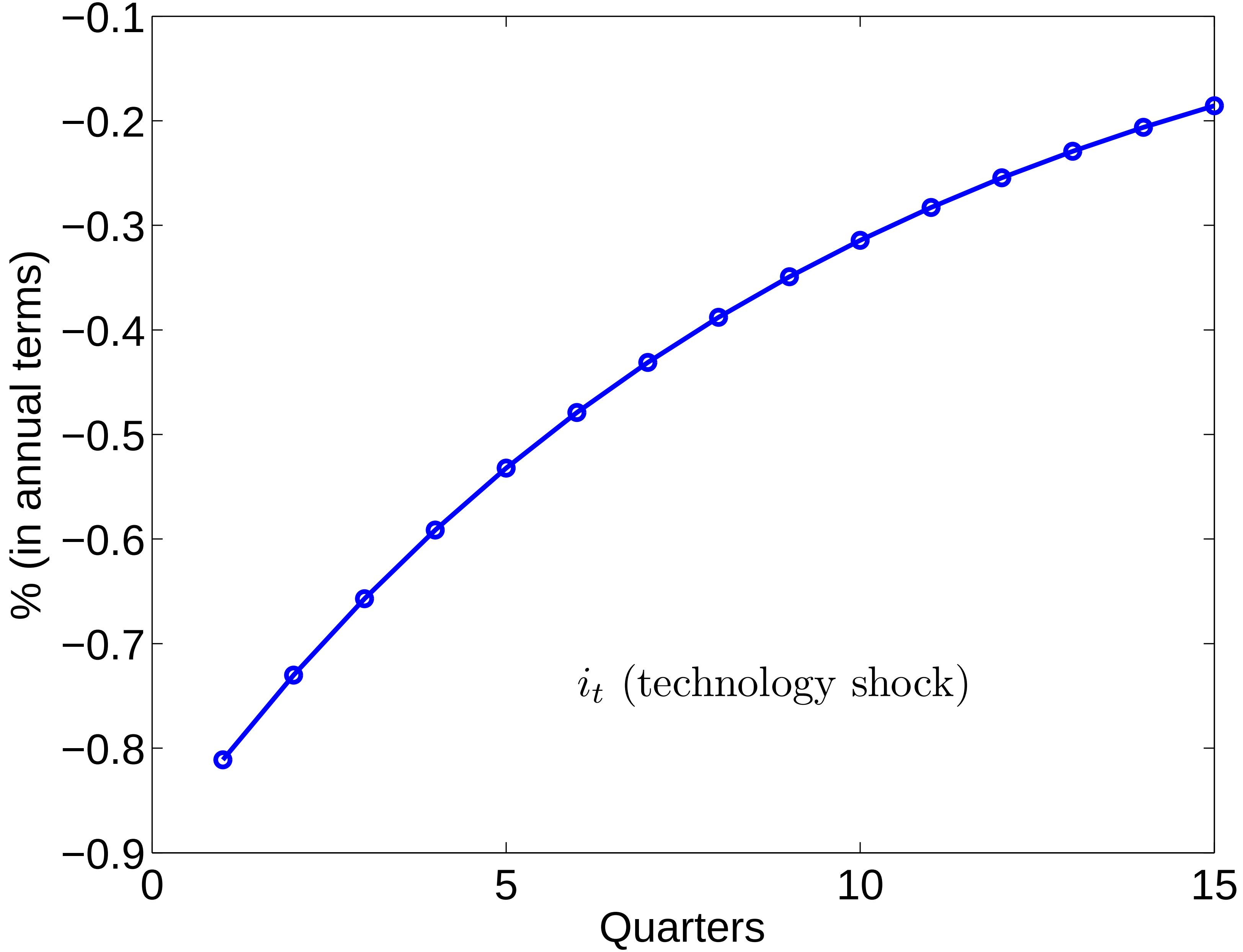
10
12
14
Quarters
r_t (technology shock)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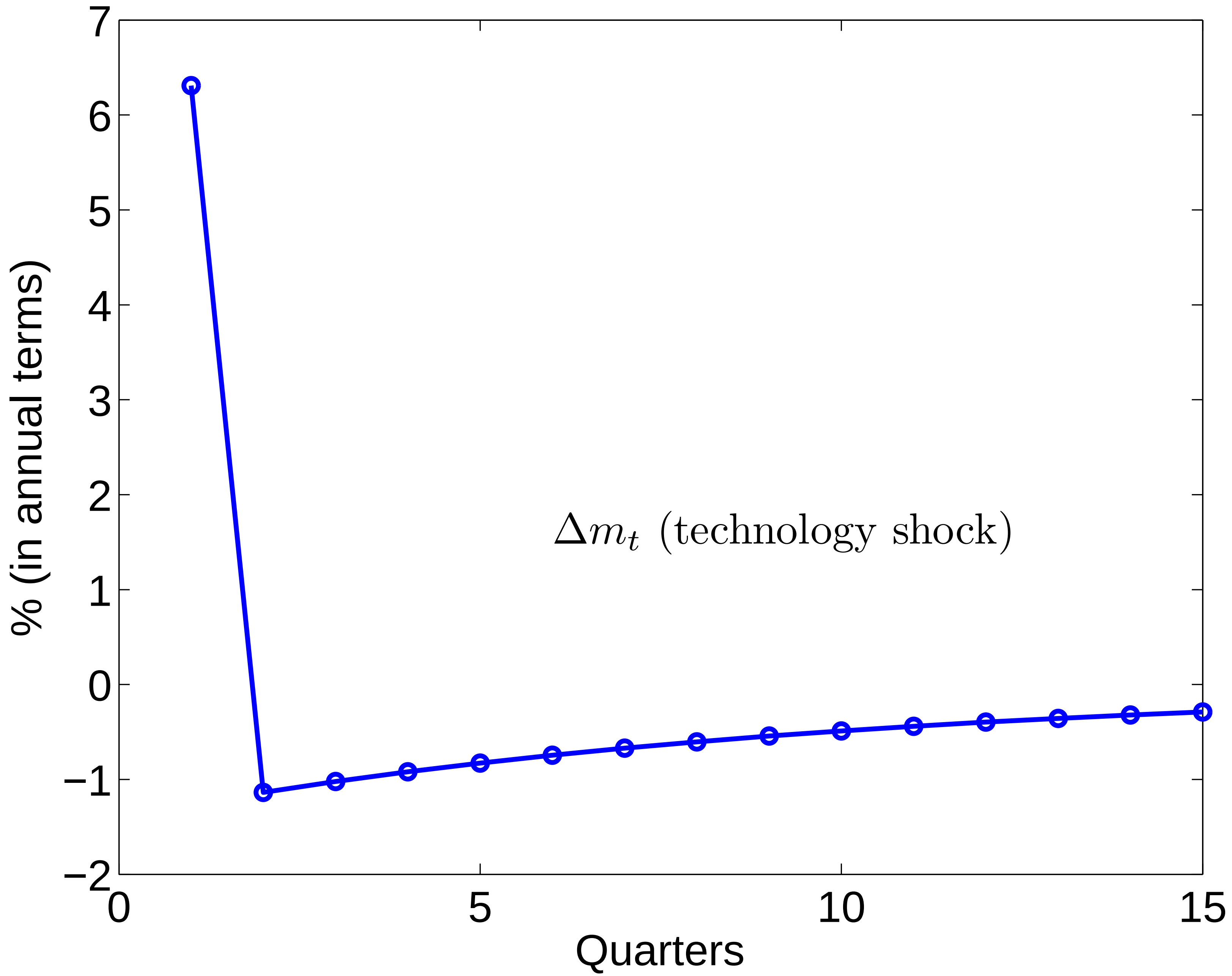
−0.2
−0.3
−0.4
−0.5
−0.6
−0.7
−0.8
−0.9
% (in annual terms)
0
5
10
15
Quarters
i_t (technology shock)


7
6
5
4
3
2
1
0
−1
−2
0
5
10
15
Quarters
% (in annual terms)
Δm_t (technology shock)

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