# Chapter 3
# Real Business Cycles

- Main Reference:

  - R. King and S. Rebelo, "Resuscitating Real Business Cycles", Handbook of Macroeconomics, 2000,

- Other references that could be read :

  - Blanchard and Fisher [1989], Chapter 7,

  - Romer [2001], Chapter 4

# 1 Introduction

The modern approach to fluctuations is presented here

• I present here the simplest version of a model that has been extensively used to model Business Cycle over the past 30 years, since Kydland & Prescott (1982).

• The main features of this model are : intertemporal general equilibrium, stochastic model, role of technological shocks

• All along, the name of the game is to reproduce some "stylized facts" of the business cycle.

- The model should be seen as illustrating a powerful tool: DSGE (Dynamic Stochastic General Equilibirum model)

# 2 Measuring the Business Cycle

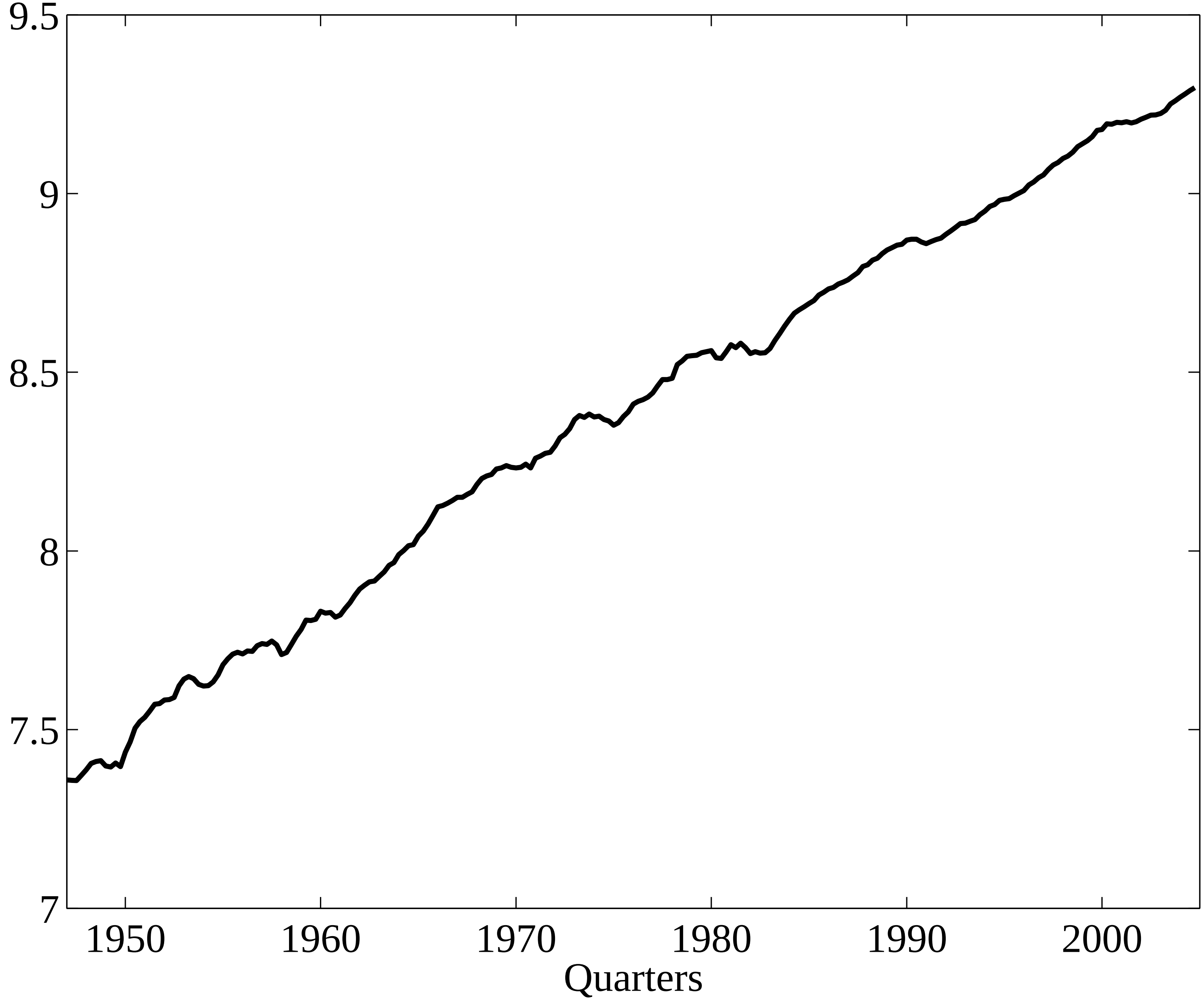
## 2.1 Trend versus Cycle

- Any Time Series can be decomposed as

$$x_t = x_t^T + x_t^c$$

- Problem: How is define/identify each component?

# Figure 1: US log GDP per capita



- Several ways of approaching the problem
- Actually: Infinite number of decomposition of a non-stationary process into a cycle and a trend
- Let us see some of those decompositions

### 2.1.1 Cycle: Output Gap

- Defined as

$$\text{Actual output} - \text{Potential Output}$$

- Expansion: Actual output $\geqslant$ Potential output
- Actual output: easy to observe
- Note: How to identify potential output? (full utilization?, efficient?)

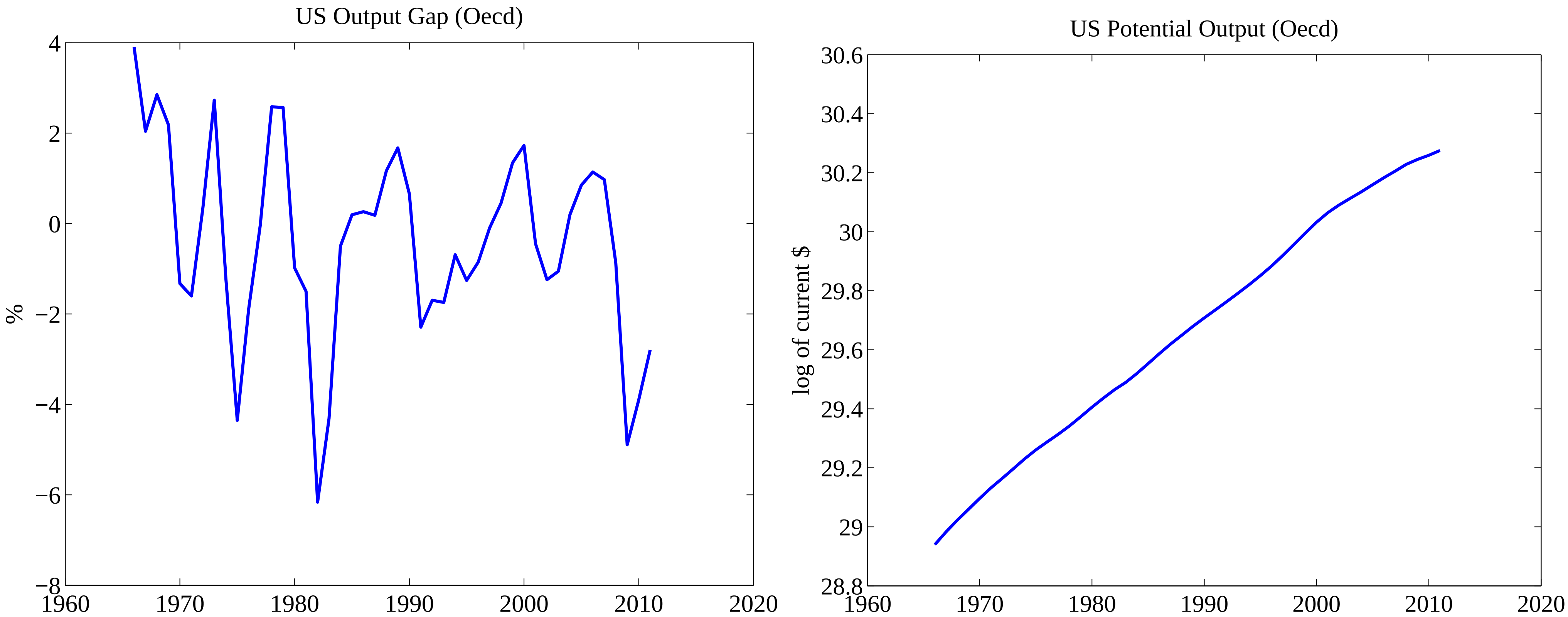
- Example:

  (1) estimate $y_t = \alpha \times u_t + \text{other controls} + \varepsilon_t$,

  (2) define potential output as $y_t^P = \widehat{\alpha}_t \times 0\% + \text{other controls} + \widehat{\varepsilon}_t$.

- This is an over simplified description of the method used by Oecd.

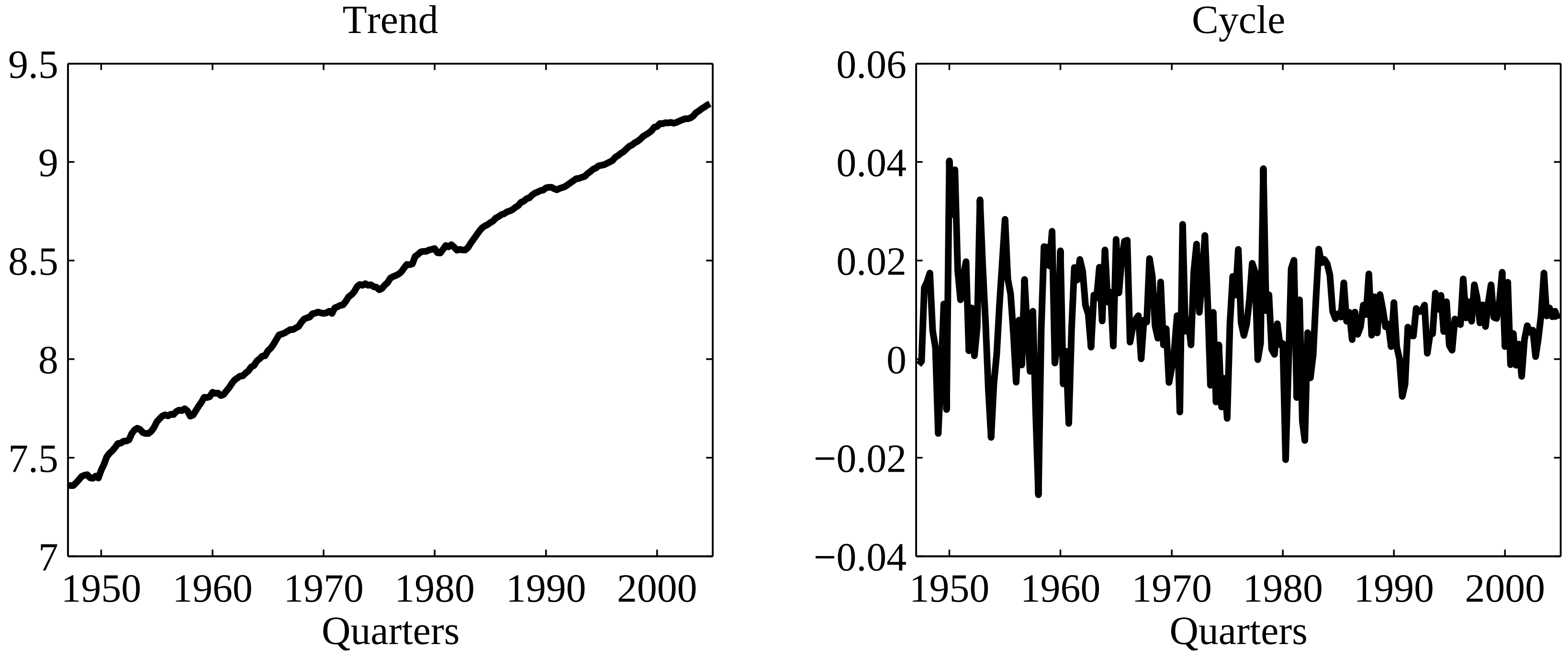
# Figure 2: US Output Gap and Potential Output



### 2.1.2 Growth Cycle

- Take the growth rate of the series
- Expansion: Positive rate of growth
- Note: the cycle is very volatile (almost iid) – a lot of medium run fluctuations are eliminated

# Figure 3: US Growth Cycles



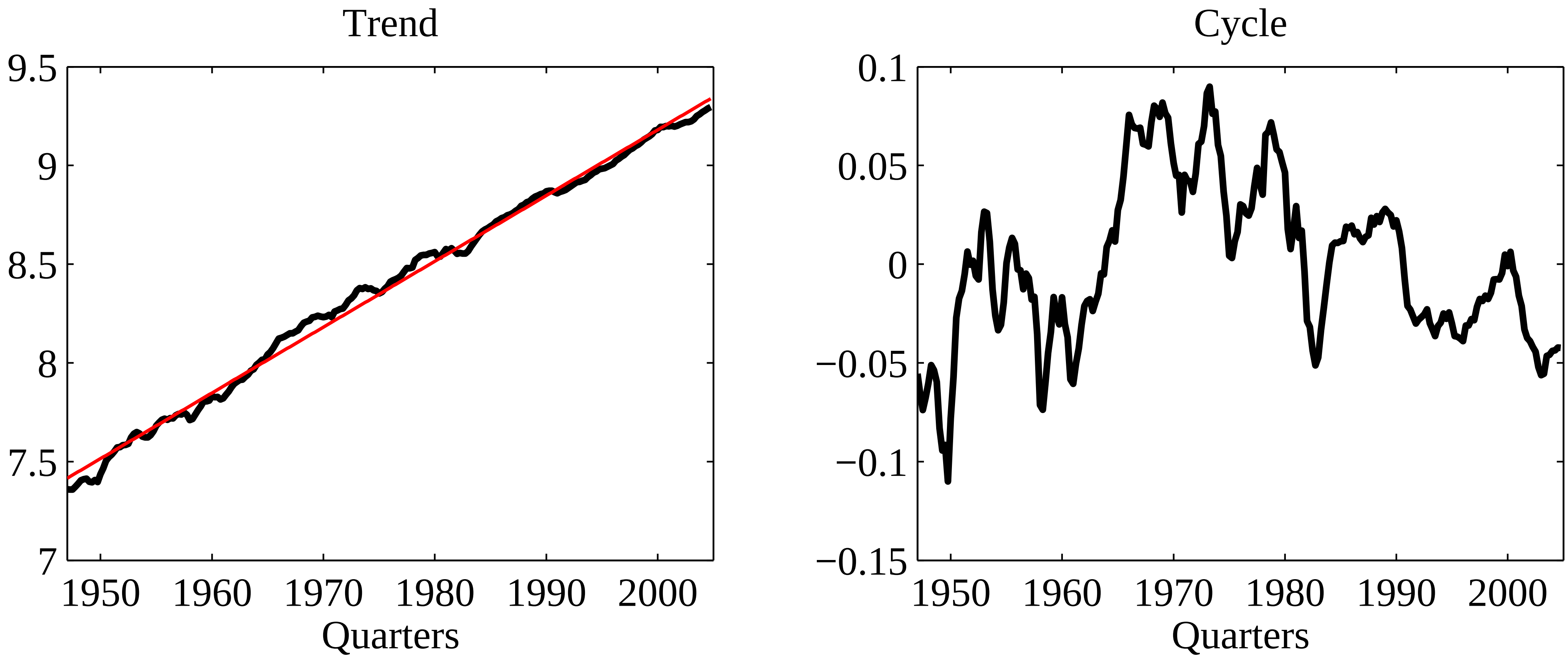
### 2.1.3 Trend Cycle

- Deviation from linear trend
- The trend is obtained from linear regression

$$\log(x_t) = \alpha + \beta t + u_t$$

- Cycle: $\widehat{x}_t = \log(x_t) - (\widehat{\alpha} + \widehat{\beta} t)$
- Expansion: Output above the trend
- Note: the cycle can be large and very persistent - a lot of medium and long run fluctuations are not eliminated

# Figure 4: US Trend Cycles



### 2.1.4 The Hodrick–Prescott Filter

- Hodrick and Prescott [1980]
- Obtained by solving

$$\min_{\{x_\tau^T\}_{\tau=1}^t} \sum_{\tau=1}^{t} \left(x_\tau - x_\tau^T\right)^2$$

subject to

$$\sum_{\tau=2}^{t-1} \left(\left(x_{\tau+1}^T - x_\tau^T\right) - \left(x_\tau^T - x_{\tau-1}^T\right)\right)^2 \leqslant c$$

- or

$$\min_{\{x_\tau^T\}_{\tau=1}^t} \sum_{\tau=1}^{t} \left(x_\tau - x_\tau^T\right)^2 + \lambda \sum_{\tau=2}^{t-1} \left(\left(x_{\tau+1}^T - x_\tau^T\right) - \left(x_\tau^T - x_{\tau-1}^T\right)\right)^2$$

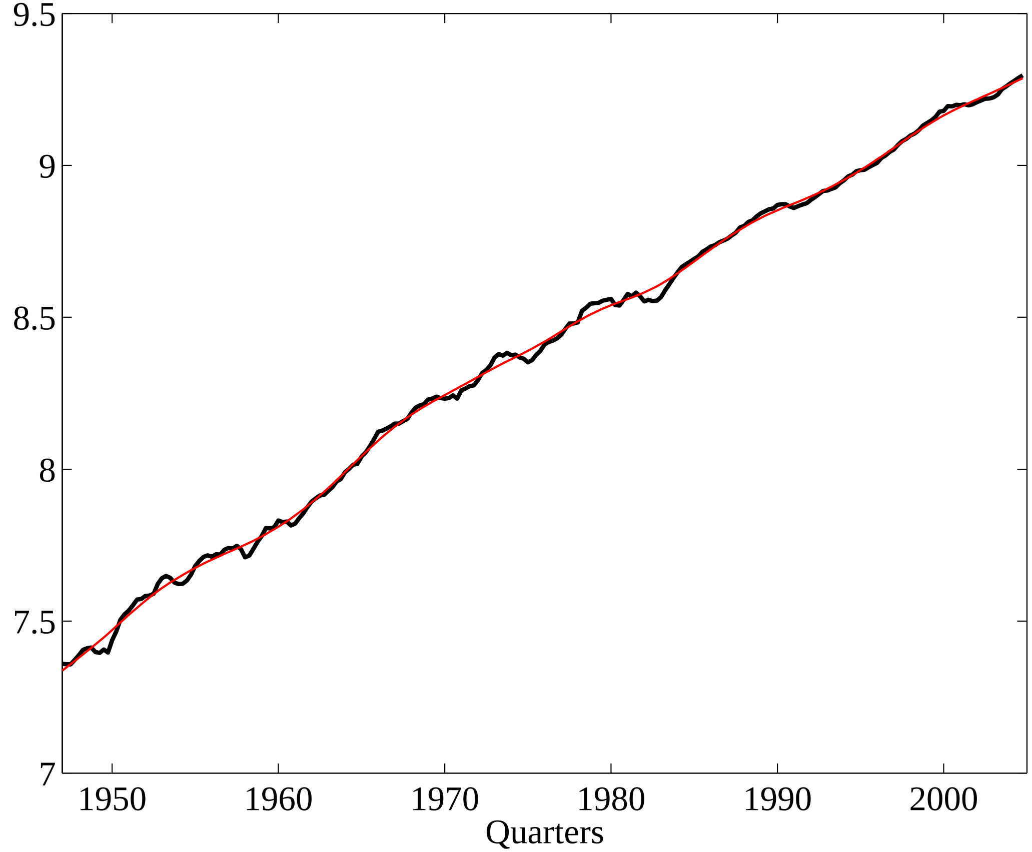
- $\lambda = 0$: the trend is equal to the series.
- $\lambda = \infty$: the trend is linear.
- Setting $\lambda$ for quarterly data: Accept cyclical variations up to 5% per quarter, and changes in the quarterly rate of growth of 1/8% per quarter, then

$$\lambda = \frac{5^2}{(1/8)^2} = 1600$$

(under some assumptions)

### 2.1.5 The HP filter at work

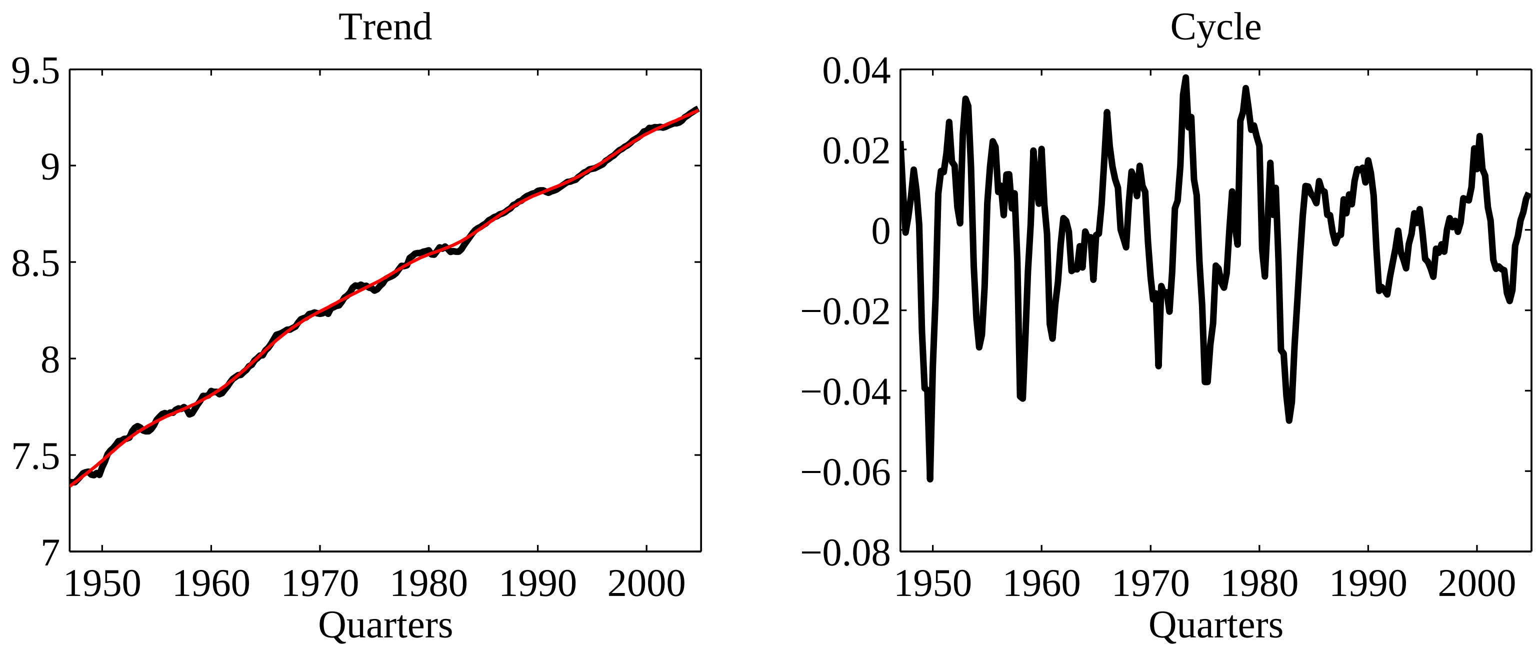
Figure 5: US HP Trend




- HP trend: not linear

- cycle is the difference between the two curves

Figure 6: US HP Cycle



- The HP filter is the one mainly used in the literature. We will use it to:

- Get the cyclical component of each macroeconomic time series,
- Compute some statistics to characterize the business cycle.

## 2.2 U.S. Business Cycles

### 2.2.1 What are Business Cycles?

- Lucas' definition:

  "Recurrent **fluctuations** of macroeconomic aggregates around trend"

- Want to find **regularities** (*Stylized facts*)

- Business Cycles are characterized by a set of statistics:

  – Volatilities of time series (standard deviations)

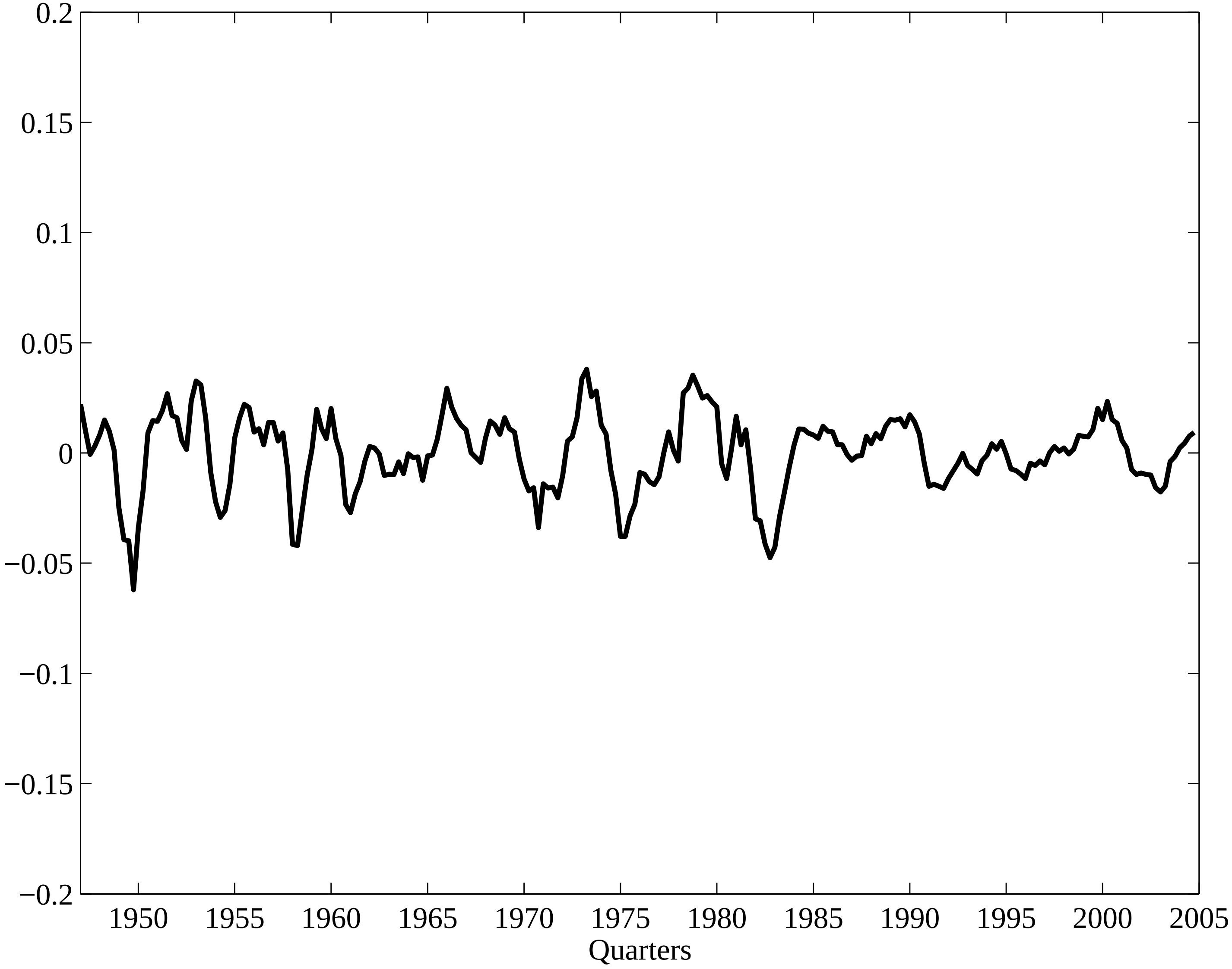
  - Comovements of time series (correlations, serial correlations)

- "*Business Cycles are all alike*"

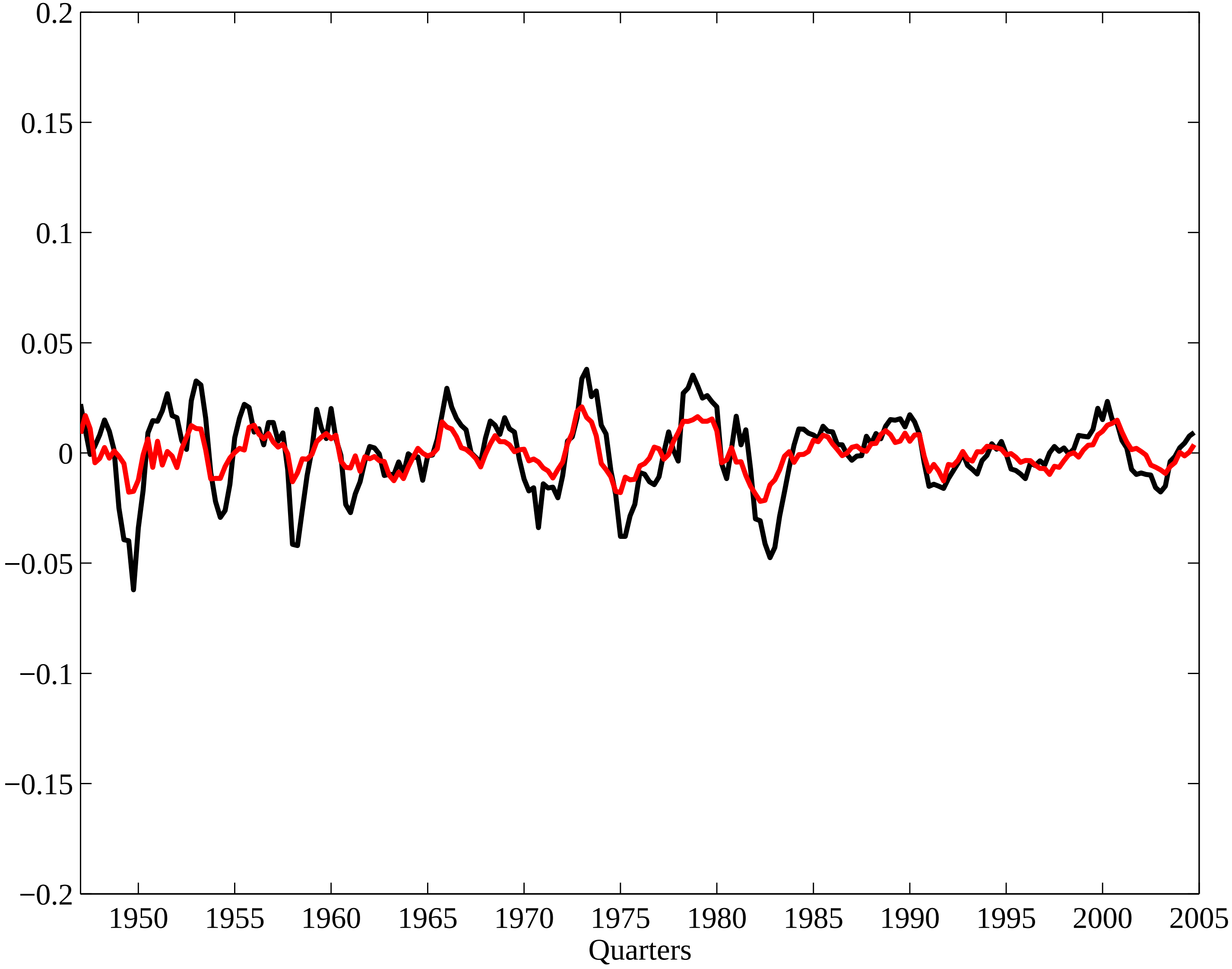
### 2.2.2 Main Real Aggregates

- Consumption ($C$): Nondurables + Services
- Investment ($I$): Durables + Fixed Investment + Changes in inventories
- Government **spending** ($G$): Absent from the basic model
- Output: $C + I + G$
- Labor: hours worked
- Labor Productivity: Output / Labor

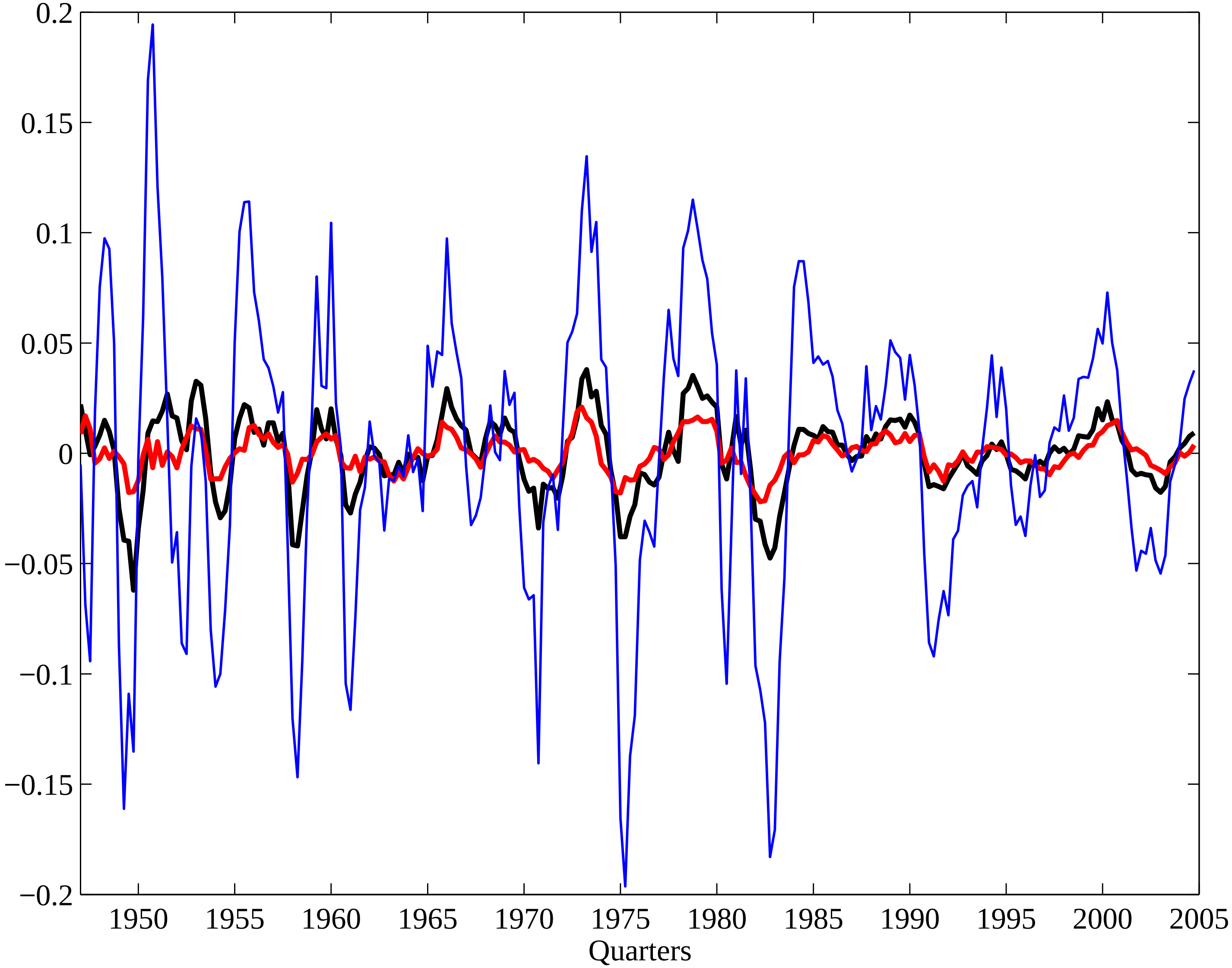
# Output


0.2
0.15
0.1
0.05
0
−0.05
−0.1
−0.15
−0.2
1950
1955
1960
1965
1970
1975
1980
1985
1990
1995
2000
2005
Quarters

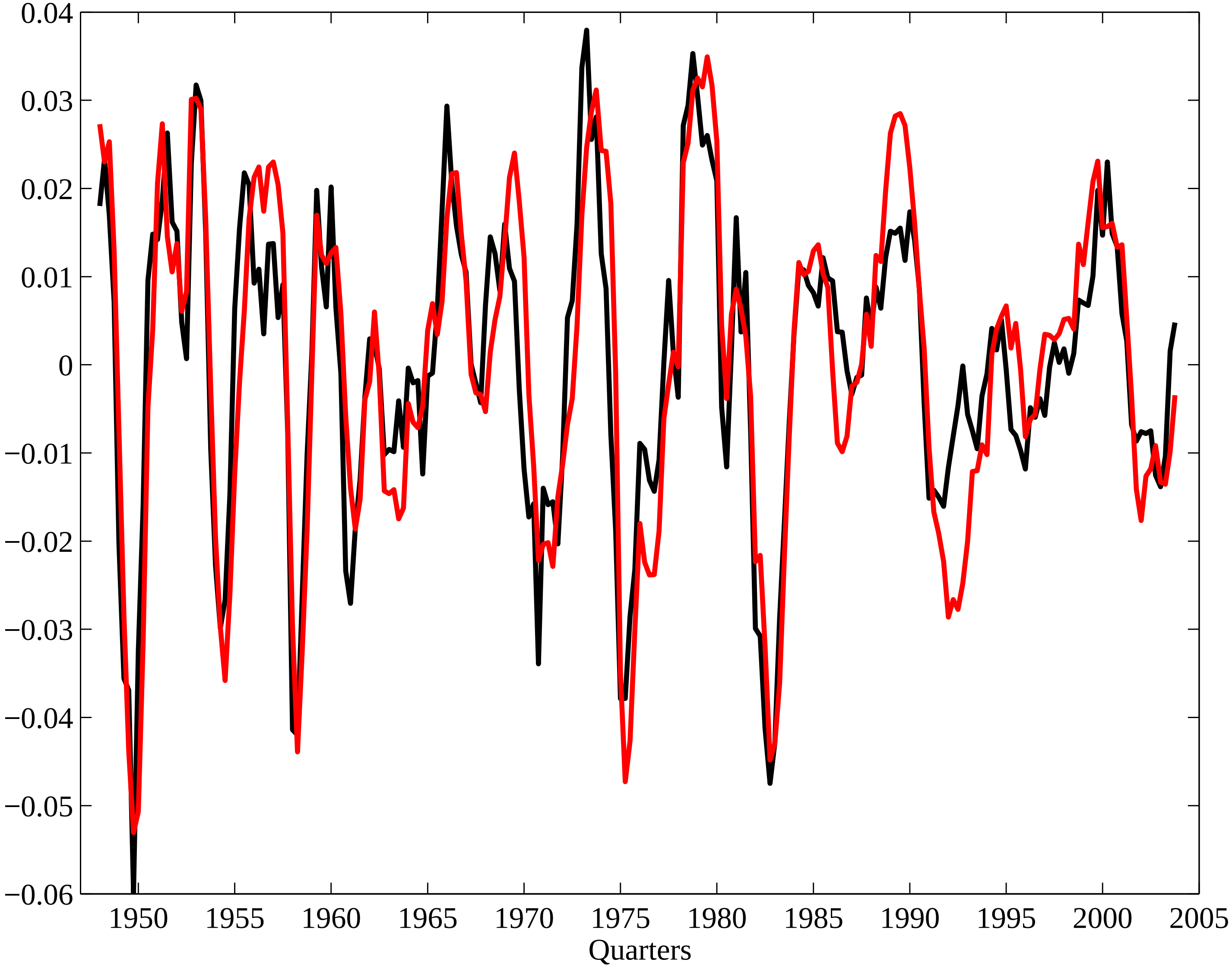
# Output – Consumption



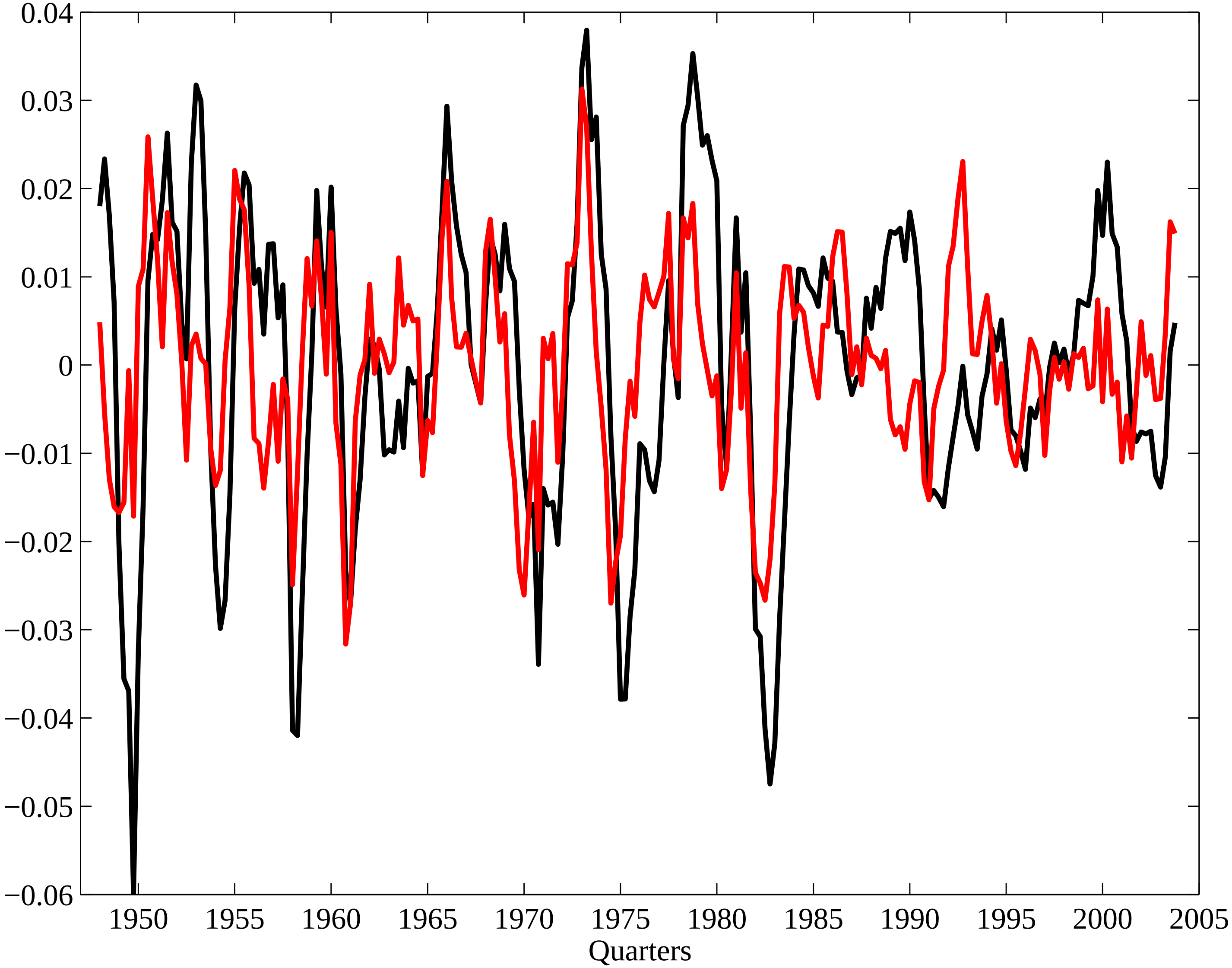
# Output – Consumption – Investment



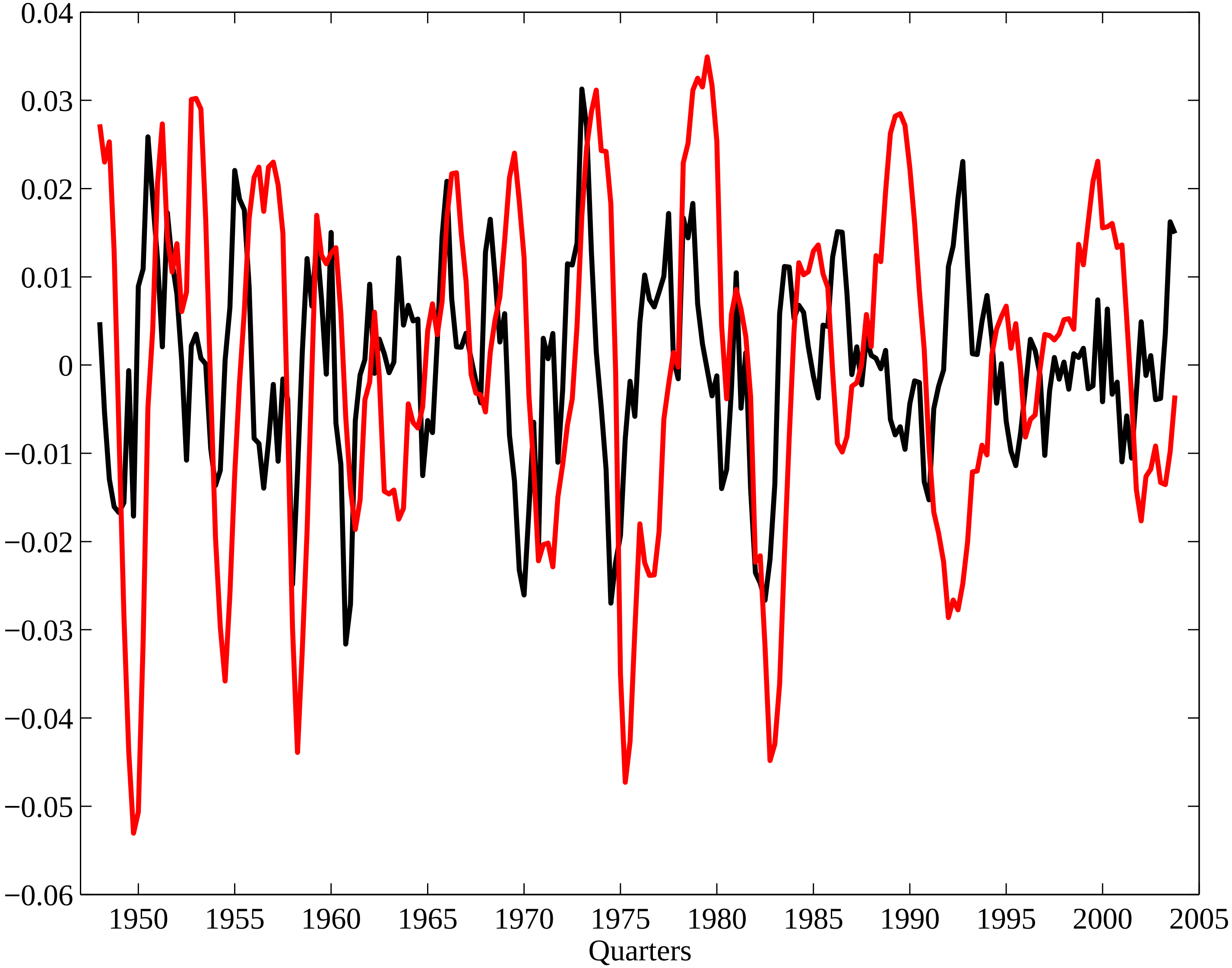
# Output – Hours worked



# Output – Productivity



# Productivity – Hours worked



### 2.2.3 Moments

- We want to characterize fluctuations $\rightsquigarrow$ amplitude and movements
- Amplitude: volatilities $\rightsquigarrow$ standard deviations
- Comovements: correlations

| Variable | $\sigma(\cdot)$ | $\sigma(\cdot)/\sigma(y)$ | $\rho(\cdot,y)$ | $\rho(\cdot,h)$ | Auto(1) |
|---|---|---|---|---|---|
| Output | 1.70 | – | – | – | 0.84 |
| Consumption | 0.80 | 0.47 | 0.78 | – | 0.83 |
| *Services* | 1.11 | 0.66 | 0.72 | – | 0.80 |
| *Non Durables* | 0.72 | 0.42 | 0.71 | – | 0.77 |
| Investment | 6.49 | 3.83 | 0.84 | – | 0.81 |
| *Fixed investment* | 5.08 | 3.00 | 0.80 | – | 0.88 |
| *Durables* | 5.23 | 3.09 | 0.58 | – | 0.72 |
| *Changes in inventories* | 22.48 | 13.26 | 0.48 | – | 0.40 |
| Hours worked | 1.69 | 1.00 | 0.86 | – | 0.89 |
| Labor productivity | 0.90 | 0.53 | 0.41 | 0.09 | 0.69 |

# Summary

1. Consumption (of non-durables) is less volatile than output
2. Investment is more volatile than output
3. Hours worked are as volatile as output
4. Capital is much less volatile than output
5. Labor productivity is less volatile than output
6. Real wage is much less volatile than output

7. All those variables are persistent and procyclical except Labor productivity that is acyclical

## 2.3 A Model to Replicate Those Facts

### The Facts

Good Market

- Consumption is less volatile than output
- Investment is more volatile than output
- Both are procyclical
- Suggests a Permanent Income component in the model

# Labor Market

- Hours is as volatile as output
- Hours are strongly procyclical
- If leisure is countercyclical, why is labor productivity high when labor is high (when we assume decreasing returns to labor)?
- Suggests that productivity shocks might drive the BC

# Towards a Model

- Needed: A macro model
- Need labor, consumption, investment ($\rightsquigarrow$ Capital)
- Dynamic model
- Can account for growth facts ($C$, $I$, $Y$ grow at the same rate in the long run)
- General **equilibrium** model

More specifically:

- Consumption $\rightsquigarrow$ Permanent income component
- Investment $\rightsquigarrow$ Capital accumulation
- Labor $\rightsquigarrow$ Labor market equilibrium
- Shocks to initiate the cycle: Technology shock

# 3 The Standard Real Business Cycle (RBC) Model

- Perfectly competitive economy
- Optimal growth model + Labor decisions
- 2 types of agents
  - Households
  - Firms
- Shocks to productivity
- Pareto optimal economy

- Can be solved using a Social Planner program or solving for a competitive equilibrium

- We will solve for the equilibrium

## 3.1 The Household

- Mass of agents = 1 (no population growth)
- Identical agents + All face the same aggregate shocks (no idiosyncratic uncertainty)
- $\rightsquigarrow$ Representative agents

- Infinitely lived rational agent with intertemporal utility

$$E_t \sum_{s=0}^{\infty} \beta^s U_{t+s}$$

  $\beta \in (0, 1)$: discount factor,

- Preferences over

  – a consumption bundle

  – leisure

- $\rightsquigarrow U_t = U(C_t, \ell_t)$ with $U(\cdot, \cdot)$

  – class $\mathcal{C}^2$, strictly increasing, concave and satisfy Inada conditions

– compatible with balanced growth [more below]:

$$U(C_t, \ell_t) = \begin{cases} \frac{C_t^{1-\sigma}}{1-\sigma} v(\ell_t) & \text{if } \sigma \in \mathbb{R}^+ \backslash \{1\} \\ \log(C_t) + v(\ell_t) & \text{if } \sigma = 1 \end{cases}$$

Preferences are therefore given by

$$E_t\left[\sum_{s=0}^{\infty}\beta^s U\big(C_{t+s},\ell_{t+s}\big)\right]$$

- Household faces two constraints

- Time constraint

$$h_{t+s}+\ell_{t+s}\leqslant T=1$$

(for convenience T=1)

- Budget constraint

$$\underbrace{B_{t+s}}_{Bond\ purchases} + \underbrace{C_{t+s} + I_{t+s}}_{Good\ purchases}$$

$$\leqslant \underbrace{(1 + r_{t+s-1})B_{t+s-1}}_{Bond\ revenus} + \underbrace{W_{t+s}h_{t+s}}_{Wages} + \underbrace{z_{t+s}K_{t+s}}_{Capital\ revenus}$$

- Capital Accumulation

$$K_{t+s+1} = I_{t+s} + (1 - \delta)K_{t+s}$$

$\delta \in (0, 1)$: Depreciation rate

The household decides on consumption, labor, leisure, investment, bond holdings and capital formation maximizing utility constraint, taking the constraints into account

$$\max_{\{C_{t+s},h_{t+s},\ell_{t+s},I_{t+s},K_{t+s+1},B_{t+s}\}_{\mathbf{s=0}}^{\infty}} E_t\left[\sum_{s=0}^{\infty}\beta^s U\big(C_{t+s},\ell_{t+s}\big)\right]$$

subject to the sequence of constraints

$$\begin{cases} h_{t+s}+\ell_{t+s}\leqslant 1 \\ B_{t+s}+C_{t+s}+I_{t+s}\leqslant \big(1+r_{t+s-1}\big)B_{t+s-1}+W_{t+s}h_{t+s}+z_{t+s}K_{t+s} \\ K_{t+s+1}=I_{t+s}+\big(1-\delta\big)K_{t+s} \\ K_t,\ B_{t-1}\text{ given} \end{cases}$$

$$\max_{\{C_{t+s},h_{t+s},K_{(+\mathbf{s}+\mathbf{1}},B_{t+s}\}_{t=0}^{\infty}} E_t \left[ \sum_{s=0}^{\infty} \beta^s U(C_{t+s}, 1-h_{t+s}) \right]$$

subject to

$$B_{t+s}+C_{t+s}+K_{t+s+1} \leqslant (1+r_{t+s-1})B_{t+s-1}+W_{t+s}h_{t+s}+(z_{t+s}+1-\delta)K_{t+s}$$

Write the Lagrangian

$$\mathcal{L}_t = E_t \textstyle\sum_{s=0}^{\infty} \beta^s \Big[ U(C_{t+s}, 1-h_{t+s}) + \Lambda_{t+s} \Big( (1+r_{t+s-1})B_{t+s-1} +$$

$$+W_{t+s}h_{t+s} + (z_{t+s}+1-\delta)K_{t+s} - C_{t+s} - B_{t+s} - K_{t+s+1} \Big) \Big]$$

# First order conditions ($\forall\ s \geq 0$)

$$
\begin{array}{ll}
C_{t+s} & : E_t U_c(C_{t+s}, 1-h_{t+s}) = E_t \Lambda_{t+s} \\
h_{t+s} & : E_t U_\ell(C_{t+s}, 1-h_{t+s}) = E_t(\Lambda_{t+s} W_{t+s}) \\
B_{t+s} & : E_t \Lambda_{t+s} = \beta E_t((1+r_{t+s})\Lambda_{t+s+1}) \\
K_{t+s+1} & : E_t \Lambda_{t+s} = \beta E_t(\Lambda_{t+s+1}(z_{t+s+1} + 1 - \delta))
\end{array}
$$

and the transversality condition

$$
\lim_{s \longrightarrow +\infty} \beta^s E_t \Lambda_{t+s}(B_{t+s} + K_{t+s+1}) = 0
$$

# Remark :

- It is convenient to write and interpret FOC for $s = 0$:

$$
\begin{aligned}
h_t &: U_\ell(C_t, 1-h_t) = E_t U_c(C_t, 1-h_t) W_t \\
B_t &: U_c(C_t, 1-h_t) = \beta E_t((1+r_t) E_t U_c(C_{t+1}, 1-h_{t+1})) \\
K_{t+1} &: U_c(C_t, 1-h_t) = \beta E_t(U_c(C_{t+1}, 1-h_{t+1})(z_{t+1} + 1 - \delta))
\end{aligned}
$$

and the transversality condition

$$\lim_{\mathbf{s} \longrightarrow +\infty} \mathbf{E_t} \beta^{\mathbf{s}} \mathbf{U_c}(\mathbf{C_{t+s}}, \mathbf{1} - \mathbf{h_{t+s}}) (\mathbf{K_{t+s+1}} + \mathbf{B_{t+s}}) = \mathbf{0}$$

**Simple example** : Assume $U(C_t, \ell_t) = \log(C_t) + \theta \log(1 - h_t)$

$$
\begin{array}{ll}
h_{t+s} & : E_t \frac{1}{1-h_{t+s}} = E_t \frac{W_{t+s}}{C_{t+s}} \\
B_{t+s} & : E_t \frac{1}{C_{t+s}} = \beta E_t (1 + r_{t+s}) \frac{1}{C_{t+s+1}} \\
K_{t+s+1} & : E_t \frac{1}{C_{t+s}} = \beta E_t \frac{1}{C_{t+s+1}} (z_{t+s+1} + 1 - \delta)
\end{array}
$$

and the transversality condition

$$\lim_{s \longrightarrow +\infty} E_t \beta^s \frac{K_{t+s+1} + B_{t+s}}{C_{t+s}} = 0$$

- We have consumption smoothing and
- We have labor smoothing

$$\frac{\theta}{W_t(1-h_t)} = \beta(1+r_t)E_t\frac{\theta}{W_{t+1}(1-h_{t+1})}$$

## 3.2 The Firm

- Mass of firms = 1
- Identical firms + All face the same aggregate shocks (no idiosyncratic uncertainty)

$\leadsto$ Representative firm

- Produce an homogenous good that is consumed or invested

- by means of capital and labor

- Constant returns to scale technology (important)

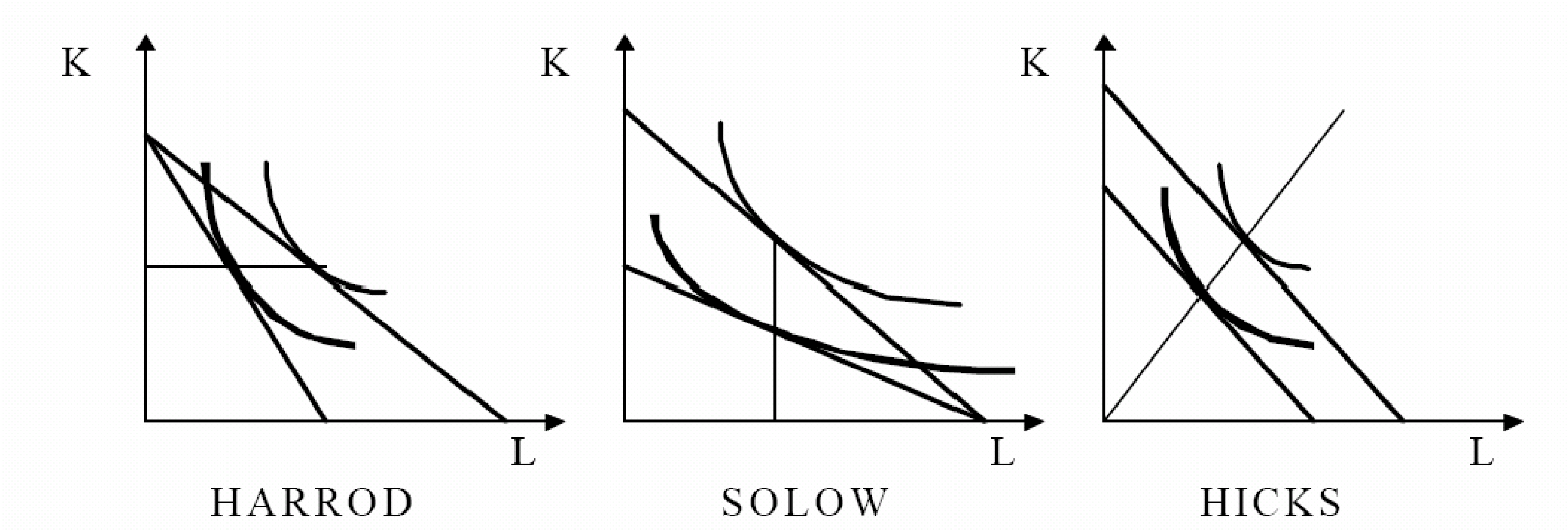
$$Y_t = A_t F(K_t, \Gamma_t h_t)$$

- $\Gamma_t = \gamma \Gamma_{t-1}$ Harrod neutral technological progress ($\gamma \geqslant 1$), $A_t$ stationary (does not explain growth)

- Remark: one could introduce long run technical progress in three different ways:

$$Y_t = \widehat{\Gamma}_t F(\widetilde{\Gamma}_t K_t, \Gamma_t h_t)$$

-$\widehat{\Gamma}_t$ is Hicks Neutral, $\widetilde{\Gamma}_t$ is Solow neutral


K
L
HARROD
K
L
SOLOW
K
L
HICKS

- Harrod neutral technical progress and the preferences specified above are needed for the existence of a *Balanced Growth Path* that replicates *Kaldor Stylized Facts*:

1. The shares of national income received by labor and capital are roughly constant over long periods of time
2. The rate of growth of the capital stock is roughly constant over long periods of time
3. The rate of growth of output per worker is roughly constant over long periods of time
4. The capital/output ratio is roughly constant over long periods of time
5. The rate of return on investment is roughly constant over long periods of time
6. The real wage grows over time

- End of the remark

$$Y_t = A_t F(K_t, \Gamma_t h_t)$$

- $\Gamma_t = \gamma \Gamma_{t-1}$ Harrod neutral technological progress ($\gamma \geqslant 1$), $A_t$ stationary (does not explain growth)

- $A_t$ are shocks to technology. AR(1) exogenous process

$$\log(A_t) = \rho \log(A_{t-1}) + (1-\rho)\log(\overline{A}) + \varepsilon_t$$

with $\varepsilon_t \leadsto \mathcal{N}(0, \sigma^2)$.

The firm decides on production plan maximizing profits

$$\max_{\{K_t,h_t\}} A_t F(K_t, \Gamma_t h_t) - W_t h_t - z_t K_t$$

First order conditions:

$$\begin{aligned} K_t &: A_t F_K(K_t, \Gamma_t h_t) = z_t \\ h_t &: A_t F_h(K_t, \Gamma_t h_t) = W_t \end{aligned}$$

**Simple Example:** Cobb–Douglas production function

$$Y_t = A_t K_t^{\alpha} (\Gamma_t h_t)^{1-\alpha}$$

First order conditions

$$\begin{aligned} K_t &: \alpha Y_t / K_t = z_t \\ h_t &: (1-\alpha) Y_t / h_t = W_t \end{aligned}$$

## 3.3 Equilibrium

The (RBC) Model Equilibrium is given by the following equations ($\forall\ t \geq 0$):

1. Exogenous Processes : $\log(A_t) = \rho \log(A_{t-1}) + (1-\rho)\log(\overline{A}) + \varepsilon_t$ and $\Gamma_t = \gamma \Gamma_{t-1}$
2. Law of motion of Capital : $K_{t+1} = I_t + (1-\delta)K_t$
3. Bond market equilibrium : $B_t = 0$
4. Good Markets equilibrium : $Y_t = C_t + I_t$

5. Labor market equilibrium : $\frac{U_\ell(C_t, 1-h_t)}{U_c(C_t, \ell_t)} = A_t F_h(K_t, \Gamma_t h_t)$

6. Consumption/saving decision + Capital market equilibrium :

$$U_c(C_t, 1-h_t) = \beta E_t \left[ U_c(C_{t+1}, 1-h_{t+1})(A_{t+1} F_K(K_{t+1}, \Gamma_{t+1} h_{t+1}) + 1 - \delta) \right]$$

7. Financial markets :

$$1 + r_t = \frac{E_t \left[ U_c(C_{t+1}, 1-h_{t+1})(A_{t+1} F_K(K_{t+1}, \Gamma_{t+1} h_{t+1}) + 1 - \delta) \right]}{E_t U_c(C_{t+1}, 1-h_{t+1})}$$

## 3.4 An Analytical Example

- $U(C_t, \ell_t) = \log(C_t) + \theta \log(\ell_t), \qquad Y_t = A_t K_t^{\alpha}(\Gamma_t h_t)^{1-\alpha}$

- Equilibrium

$$\frac{\theta C_t}{1-h_t} = (1-\alpha)\frac{Y_t}{h_t}$$

$$\frac{1}{C_t} = \beta E_t \left[\frac{1}{C_{t+1}}\left(\frac{Y_{t+1}}{K_{t+1}} + 1 - \delta\right)\right]$$

$$K_{t+1} = Y_t - C_t + (1-\delta)K_t$$

$$Y_t = A_t K_t^{\alpha}(\Gamma_t h_t)^{1-\alpha}$$

$$\lim_{s \longrightarrow \infty} \beta^s E_t \left[\frac{K_{t+1+s}}{C_{t+s}}\right] = 0$$

## 3.5 Stationarization

- We want a stationary equilibrium (technical reasons)
- Deflate the model for the growth component $\Gamma_t$
- On the example: $x_t = X_t / \Gamma_t$

- Deflated Equilibrium

$$\frac{\theta c_t}{1-h_t} = (1-\alpha)\frac{y_t}{h_t}$$

$$\frac{1}{c_t} = \frac{\beta}{\gamma} E_t\left[\frac{1}{c_{t+1}}\left(\frac{y_{t+1}}{k_{t+1}} + 1 - \delta\right)\right]$$

$$\gamma k_{t+1} = y_t - c_t + (1-\delta)k_t$$

$$y_t = A_t k_t^{\alpha} h_t^{1-\alpha}$$

$$\lim_{s \longrightarrow \infty} \beta^s \frac{\gamma k_{t+1+s}}{c_{t+s}} = 0$$