1 Introduction

These notes summarize the Solow growth model. The Solow growth model provides a relatively simple framework from which we can gain some basic insight into the determinants of living standards and how an economy might grow over time.

The main goal of these notes are to:

1. Understand the long-run determinants of living standards.
2. Study transition dynamics of an economy within the context of a neoclassical framework.

2 Preliminary material 1: Production structure

The Solow growth model requires us to be specific about the production structure of the economy, the process by which factor inputs such as capital and labor evolve, and the process by which technology evolves. We begin by describing the production structure. We then consider the evolution of both labor and technology, after which we consider the savings behavior and the process of capital accumulation. With these features in place, we are then in a position to describe how the economy evolves over time.

2.1 The production function

The production structure specifies the relationship between inputs such as capital and labor, the available technology, and the amount of output that can be produced from such inputs and
technology. This relationship is summarized by a production function — a function relating inputs to output. Neoclassical economics implies that production functions should exhibit diminishing returns to each input. We will further assume that the production function has the property of constant returns to scale. This assumption is both realistic and convenient — it allows us to consider a normalization of the production function that will provide a simple characterization of the evolution of key economic variables over time.

We assume that the economy produces a single good $Y_t$ by means of two factors of production – capital, $K_t$ and labor, $L_t$. The productivity of these inputs also depends on the level of technology $A_t$. $Y_t$ should be understood as macroeconomic value-added, i.e. The production of good and services net of intermediate inputs. It corresponds to Gross Domestic Product in the data. We assume that $A_t$ is “labor augmenting”: An increase in $A_t$ improves the effectiveness of workers in production. More specifically, we let $A_t L_t$ denote the efficiency units of labor for a given labor input $L_t$ and technology $A_t$. The production function may be written as

$$Y_t = F(K_t, A_t L_t).$$

In general, it is reasonable to assume that increasing either capital or efficiency units of labor leads to an increase in output. Thus the marginal product of capital and the marginal product of labor is positive. Formally, the marginal product is defined as the increment to output obtained from hiring one more unit of an input, holding all other inputs and technology constant. Mathematically, the marginal product of an input is simply the partial derivative of the production function with respect to each input.

The marginal product of capital is:

$$MPK_t = \frac{\partial F(K_t, A_t L_t)}{\partial K_t}.$$

The marginal product of labor is:

$$MPL_t = \frac{\partial F(K_t, A_t L_t)}{\partial L_t}.$$

The statement that output is increasing in either capital or labor is equivalent to saying that the marginal products are positive:

$$MPK_t, MPL_t > 0.$$
As an illustration of these ideas, consider the following “Cobb-Douglas” production function:

\[ Y_t = K_t^\alpha (A_t L_t)^{1-\alpha} \]

where \( \alpha \in [0, 1] \).

To compute the marginal product of capital for this production function, we want to take the partial derivative of \( Y_t \) with respect to \( K_t \). Taking the partial derivative is equivalent to taking the derivative of \( Y_t \) with respect to \( K_t \) holding \((A_t L_t)^{1-\alpha}\) fixed. We then have

\[ \frac{\partial Y_t}{\partial K_t} = \alpha K_t^{\alpha-1} (A_t L_t)^{1-\alpha} = \alpha \left( \frac{Y_t}{K_t} \right) \]

This equation states that, for the Cobb-Douglas production function, the marginal product of capital is proportional to the average product of capital \((Y_t/K_t)\). The factor of proportionality, \( \alpha \), is less than one, thus the marginal product is strictly less than the average product.

To compute the marginal product of labor we proceed in a similar manner except that we should recognize that an increase in labor increases the efficiency units of labor \((A_t L_t)\) which in turns leads to an increase in output. To compute the marginal product of labor, it is easiest to proceed by first computing the effect of an increase in efficiency units of labor on output, and then taking into account the effect of an increase in labor on efficiency units.

The partial derivative of output with respect to labor is:

\[ \frac{\partial Y_t}{\partial L_t} = (1-\alpha) K_t^\alpha A_t^{1-\alpha} L_t^{-\alpha} = (1-\alpha) \left( \frac{Y_t}{L_t} \right) \]

So the marginal product of labor is proportional to the average product of labor \((Y_t/L_t)\) with factor of proportionality \((1-\alpha) < 1\).

### 2.1.1 Disminating returns

The principal of diminishing returns implies that the incremental increase in output obtained from adding one more unit of an input is decreasing in the amount of the input already hired. Thus, the marginal product of hiring one more unit of capital when \( K_t = 1 \) should be higher than the marginal product of capital when \( K_t = 2 \) for example. Mathematically, diminishing returns is equivalent to saying that the derivative of the marginal product of an input is negative (i.e. at higher levels of the input, the marginal product will be lower):

\[ \frac{\partial MPP_{K_t}}{\partial K_t} < 0, \]
\[
\frac{\partial MPL_t}{\partial L_t} < 0.
\]

For our Cobb-Douglas example, taking second derivatives implies

\[
\frac{\partial MPK_t}{\partial K_t} = -\alpha (1 - \alpha) \frac{Y_t}{K_t^2} < 0,
\]
\[
\frac{\partial MPL_t}{\partial L_t} = -\alpha (1 - \alpha) \frac{Y_t}{L_t^2} < 0,
\]

2.1.2 Constant returns to scale

In addition to diminishing returns, we also assume that the production function exhibits constant returns to scale. A production function exhibits constant returns to scale, if doubling both inputs implies that output also doubles (holding technology fixed). More generally we say that the production function has constant returns to scale if

\[
F (\lambda K_t, \lambda A_t L_t) = \lambda F (K_t, A_t L_t)
\]

for some factor of proportionality \(\lambda > 0\).

Now consider increasing each input by the proportion \(\lambda\) in our Cobb-Douglas production function:

\[
Y_1 = F (\lambda K_t, \lambda A_t L_t) = (\lambda K_t)^\alpha (A_t \lambda L_t)^{1-\alpha}
\]
\[
= \lambda K_t^\alpha (A_t L_t)^{1-\alpha} = \lambda Y_0 = \lambda F (K_t, A_t L_t).
\]

2.1.3 Some technicalities

In addition to diminishing returns to each input and constant returns to scale, we assume the following “technical” conditions on production. Each input strictly is necessary:

\[
F (0, L_t) = F (K_t, 0) = 0
\]

\(F (K_t, 1)\) is strictly concave and

\[
\lim_{K_t \to 0} \frac{\partial F}{\partial K_t} (K, 1) = \infty,
\]
\[
\lim_{K_t \to \infty} \frac{\partial F}{\partial K_t} (K, 1) = 0.
\]

These conditions guarantee a positive level of per-capita output in the long-run.
2.2 The intensive form of the production function

Given inputs $K_t, L_t$ and technology $A_t$, output is

$$Y_t = F(K_t, A_t L_t)$$

Constant returns to scale implies that if we divide all inputs by $(A_t L_t)$ we end up with $(Y_t/A_t L_t)$ units of output:

$$\frac{Y_t}{A_t L_t} = F\left(\frac{K_t}{A_t L_t}, 1\right)$$

$$y_t = F(k_t, 1)$$

where $y_t$ denotes output per effective unit of labor, which depends only on capital per effective unit of labor $k_t$.

Let $f(k_t) \equiv F(k_t, 1)$ denote the ”intensive form” of the production function. It may be checked that $f(k_t)$ has the same properties as $F(k_t, 1)$:

$$f'(k_t) = \frac{\partial F(k_t, 1)}{\partial k_t} > 0,$$

$$f''(k_t) = \frac{\partial^2 F(k_t, 1)}{\partial k_t^2} < 0,$$

and

$$\lim_{k_t \to 0} f'(k_t) = \infty,$$

$$\lim_{k_t \to \infty} f'(k_t) = 0.$$  

The intensive form of the production function determines how output per effective unit of labor varies with the amount of capital per effective unit of labor that is used as an input. As the amount of capital per effective unit of labor increases, output per effective unit of labor also increases but at a diminishing rate.

3 Preliminary material 2: Growth rates in discrete vs continuous time

In this section we briefly review the relationship between growth rates, and log-differences for both discrete and continuous time. For a variable that evolves continuously over time, we let $X(t)$
denote the variable $X$ at time $t$. For a variable that evolves in discrete time (period-by-period), we let $X_t$ denote the variable at date $t$ (e.g. a year, quarter etc.). Suppose that the variable $X_t$ is growing at a constant rate $g$ over time, then

$$X_t = (1 + g)X_{t-1}$$

or

$$\Delta\%X_t = \frac{X_t - X_{t-1}}{X_{t-1}} = g$$

where $g$ denotes the percentage change in $X$. For a variable that starts at value $X_0$ at time 0, after $t$ periods we have:

$$X_t = (1 + g)^t X_0$$

owing to compounding.

Going back to the one period expression and taking logs we have:

$$\log X_t = \log (1 + g) + \log X_{t-1},$$

so that

$$\log X_t - \log X_{t-1} = \log (1 + g).$$

For small values of $g$, we can use the following approximation:

$$\log (1 + g) \approx g.$$

Let $\Delta \log X_t = \log X_t - \log X_{t-1}$. Then the log-difference in $X_t$ is the growth rate:

$$\Delta \log X_t \approx g,$$

so that

$$\Delta \log X_t \approx \Delta\%X_t.$$

Log differences are equivalent to percent changes. In fact, throughout the course, we will use log differences as our measure of percent changes. Similarly, taking logs of our expression for the evolution of $X_t$ after $t$ periods we have

$$\log X_t = t \log (1 + g) + \log X_0,$$
or, using our approximation

\[ \log X_t \approx gt + \log X_0. \]

For a variable that evolves in discrete time, this expression is an approximation. As the length of the time period becomes smaller, this approximation becomes more accurate. To see this, let's now consider a variable that evolves in continuous time.

In the discrete time case we considered an evolution equation for the variable \( X_t \) of the form \( X_t = (1 + g) X_{t-1} \), where \( g \) is the growth rate over the period. Rearranging this equation we have:

\[ \frac{X_t - X_{t-1}}{X_{t-1}} = \frac{\Delta X_t}{\Delta t} \frac{1}{X_{t-1}} = g \]

where \( \Delta t \) denotes the time period over which the growth rate is computed (e.g. a year). Rearranging this equation we have an equation for the evolution of \( X \) over time in the following form:

\[ \frac{\Delta X_t}{\Delta t} = g X_{t-1}. \]

As we shrink the time period that we consider to zero, we say that \( X_t \) evolves in "continuous time". To make the distinction between discrete and continuous time transparent we often adopt the notation \( X(t) \) to denote the idea that \( X \) is a continuous function of time \( t \). We then interpret \( \lim_{\Delta t \to 0} \frac{\Delta X_t}{\Delta t} \) as the instantaneous change in \( X \) at time \( t \). This is equivalently, the derivative of \( X \) with respect to time:

\[ \lim_{\Delta t \to 0} \frac{\Delta X_t}{\Delta t} = \frac{dX(t)}{dt}. \]

Our percentage change equation then becomes

\[ \frac{dX(t)}{dt} \frac{1}{X(t)} = g \]

or

\[ \frac{dX(t)}{dt} = gX(t). \]

It is natural to consider this equation as a description of how \( X(t) \) evolves in continuous time. We can obtain such an equation by assuming that

\[ X(t) = X(0) \exp(gt) \]

in which case, after taking derivatives we recover

\[ \frac{dX(t)}{dt} = gX(0) \exp(gt) = gX(t). \]
Note we could also take logs of $X(t)$:

$$\log (X(t)) = \log (X(0)) + gt$$

in which case

$$\frac{d \log X(t)}{dt} = g.$$ 

Over small time periods, the percentage change in a variable $X(t)$ is approximately equal to the change in the log of $X(t)$. For a continuous-time variable, this relationship is exact.

## 4 The Solow Growth Model

We now turn to studying the Solow growth model. Our goal is to provide a relatively simple description of the process of capital accumulation and growth that is nonetheless rich enough to provide some useful insight into the determinants of living standards, and the evolution of living standards over time (transition dynamics). In the Solow growth framework, we distinguish between exogenous variables which are determined outside of the model, and endogenous variables that respond to forces within the model. In particular, the Solow growth model assumes that the growth rate of the population and the growth rate of technology are exogenously given. It also assumes that the savings rate - the fraction of output that is saved — is determined exogenously. In contrast, the economy’s levels of savings, investment, capital and output are determined endogenously, i.e. within the model. We can then consider the implications of changes in the savings rate or changes in the growth rates of technology and labor for capital accumulation, output and living standards.

### 4.1 Model Set-up

#### 4.1.1 Production

Following our discussion of production functions above we characterize output in the economy as a function of factors of production capital and labor, and an index of technology. Let $Y(t)$ denote output produced at time $t$ with capital $K(t)$ and labor $L(t)$. Production in this economy satisfies

$$Y(t) = F(K(t), A(t) L(t))$$
where $A(t)$ denotes the index of technological efficiency and $A(t)L(t)$ measures effective labor inputs. The production function $F(K, AL)$ displays diminishing returns to both inputs:

\[
F_K > 0, \quad F_{KK} < 0
\]

\[
F_L > 0, \quad F_{LL} < 0
\]

and constant return to scale in capital and labor:

\[
F (\lambda K (t), A (t) \lambda L (t)) = \lambda F (K (t), A (t) L (t))
\]

for any $\lambda > 0$.

### 4.1.2 Evolution of labor and technology

Starting from initial values $L(0)$ and $A(0)$, we assume that labor and technology grow at constant rates

\[
A (t) = A (0) \exp (gt),
\]

\[
L (t) = L (0) \exp (nt),
\]

where $g$ denotes the growth rate in technology and $n$ denotes the growth rate in population.

### 4.1.3 Investment and savings

In a discrete time model, capital accumulation is assumed to be a function of past undepreciated capital plus gross investment:

\[
K_{t+1} = (1 - \delta) K_t + I_t
\]

where $\delta < 1$ denotes the depreciation rate. Taking differences per unit of time we have:

\[
\frac{\Delta K_{t+1}}{\Delta t} = I_t - \delta K_t.
\]

Taking the limit $\Delta t \to 0$ provides the continuous time equation for the evolution of capital:

\[
\frac{dK(t)}{dt} = I (t) - \delta K (t)
\]

where $\delta$ denotes the instantaneous rate of depreciation and $I(t)$ denotes gross investment at time $t$. Let $C(t)$ denote total consumption and $S(t)$ denote economy-wide savings. We assume that output in the economy is either consumed or saved:

\[
Y (t) = C (t) + S (t).
\]
We also assume that a constant fraction of output is saved each period:

\[ S(t) = sY(t) \]

and investment equals savings so that

\[ I(t) = sY(t). \]

Combining these expressions we have

\[ \frac{dK(t)}{dt} = sY(t) - \delta K(t). \]

### 4.1.4 Model summary

Labor and technology evolve exogenously according the exponential growth equations:

\[ A(t) = A(0) \exp(gt), \]
\[ L(t) = L(0) \exp(nt). \]

The capital stock depends on savings which is proportional to output, and undepreciated capital:

\[ \frac{dK(t)}{dt} = sY(t) - \delta K(t). \]

Output depends on technology and factor inputs:

\[ Y(t) = F(K(t), A(t)L(t)). \]

The growth rates of labor and technology, \( g, n \), along with savings rate are exogenously given parameters in the model. We would now like to characterize how this economy evolves over time for a given set of initial conditions \( L(0), A(0), K(0) \).

### 4.2 Model solution

To characterize the model solution we will normalize all variables by effective units of labor. Capital per effective unit of labor is defined as:

\[ k(t) = \frac{K(t)}{A(t)L(t)}. \]
Output per effective unit of labor is

\[ y(t) = \frac{Y(t)}{A(t)L(t)} = f(k(t)) \]

where

\[ f(k(t)) = F\left(\frac{K(t)}{A(t)L(t)}, 1\right) \]

is the “intensive form” of the production function. Dividing the capital accumulation equation by \( A(t)L(t) \) we then have

\[ \frac{dK(t)}{dt} \frac{1}{A(t)L(t)} = sy(t) - \delta k(t). \]

### 4.2.1 Evolution of capital per effective unit of labor

To solve the model, it is sufficient to describe the evolution of capital per effective unit of labor \( k(t) \). To do so, we must first take the time derivative of \( k(t) = K(t)/(A(t)L(t)) \). Note that it is made up of several terms, we use standard calculus techniques to obtain

\[ \frac{dk(t)}{dt} = \frac{1}{A(t)L(t)} \frac{dK(t)}{dt} - \frac{K(t)}{(A(t)L(t))^2} \left[ L(t) \frac{dA(t)}{dt} + A(t) \frac{dL(t)}{dt} \right] \]

Using the facts that \( \frac{dA(t)}{dt} = gA(t) \) and \( \frac{dL(t)}{dt} = nL(t) \), we have

\[
\begin{align*}
\frac{dk(t)}{dt} &= \frac{1}{A(t)L(t)} \frac{dK(t)}{dt} - \frac{K(t)}{(A(t)L(t))^2} \left[ L(t) gA(t) + A(t) nL(t) \right] \\
&= \frac{1}{A(t)L(t)} \frac{dK(t)}{dt} - \frac{K(t)}{A(t)L(t)} [g + n] \\
&= \frac{1}{A(t)L(t)} \frac{dK(t)}{dt} - k(t) [g + n]
\end{align*}
\]

From the capital accumulation equation we have:

\[ \frac{1}{A(t)L(t)} \frac{dK(t)}{dt} = sy(t) - \delta k(t) \]

Plugging this equation in our expression for \( \frac{dk(t)}{dt} \), we obtain the equation describing the evolution of capital per effective unit of labor:

\[ \frac{dk(t)}{dt} = sf(k(t)) - (g + n + \delta) k(t). \]
4.3 Model dynamics

The evolution for $k(t)$ satisfies:

$$\frac{dk(t)}{dt} = sf(k(t)) - (g + n + \delta)k(t).$$

The first term in this expression is actual savings (or investment) per effective unit of labor. The second term in this expression is the "required investment" to maintain a constant capital per effective unit of labor $k(t)$. In other words, to maintain a constant $k(t)$, investment per effective unit of labor must cover growth in the economy owing to both technology and population growth $(g + n)k(t)$ and depreciation of existing capital $\delta k(t)$.

When actual investment $sf(k(t))$ exceeds required investment, $k(t)$, is growing. When actual investment is below required investment, $k(t)$ is decreasing.

At the point,

$$sf(k(t)) = (g + n + \delta)k(t)$$

the economy is in steady-state with a constant $k(t) = k^*$. Because $f(k(t))$ is concave, it is easy to show that

$$\frac{dk(t)}{dt} > 0 \text{ for } k(t) < k^*$$

and

$$\frac{dk(t)}{dt} < 0 \text{ for } k(t) > k^*$$

In words, if capital is below the steady-state value, $k(t)$ is increasing. If capital is above the steady-state value, $k(t)$ is decreasing. These results are also apparent from the Figure 4.3 next page, since, when $k(t) < k^*$ is low, $sf(k(t)) > (g + n + \delta)k(t)$ and investment exceeds the break-even amount (the green line is above the blue line) and vice versa. This implies stable dynamics—regardless of where we start, we will eventually converge to the steady-state value $k^*$.

The Figure 4.3 depicts the relationship between the change in $k(t)$ and the level of $k(t)$. For $k(t) < k^*$ we have $dk(t)/dt > 0$ and for $k(t) > k^*$ we have $dk(t)/dt < 0$. 


Figure 1: Steady state capital level $k^*$

Figure 2: Dynamics of $k(t)$
4.4 Steady state

The steady-state of this economy occurs at the point $k^*$ where

$$sf \left( k^* \right) = \left( g + n + \delta \right) k^*$$

Because $f(k)$ is concave, it is easy to verify that an increase in $s$ must cause an increase in $k^*$. These results may be seen by plotting the investment line, $sf(k)$ and the required investment line $(g+n+\delta)k$ on the same graph and examining the effect of a change in $s$ on the point of intersection of these two lines. Similarly, an increase in the term $(g + n + \delta)$ causes a reduction in $k^*$. This is Figure 4.4.

To obtain an algebraic solution for the steady-state value of $k(t)$ we must specify a production function. With Cobb-Douglas production,

$$Y(t) = K(t)^\alpha \cdot (A(t) \cdot L(t))^{1-\alpha}$$

and in intensive form we have

$$f(k) = k^\alpha$$

where $\alpha \in [0, 1]$. The steady state $k^*$ satisfies

$$s \left( k^* \right)^\alpha = \left( g + n + \delta \right) k^*.$$
Solving for \( k^* \) we have
\[
k^* = \left( \frac{s}{g + n + \delta} \right)^{\frac{1}{1-\alpha}}.
\]
From this expression, we can readily see that an increase in the savings rate \( s \) will raise \( k^* \) while an increase in \( g + n + \delta \) will lower \( k^* \).

With the Cobb-Douglas case, we can also calculate the effect of an increase in savings on output. Note that
\[
y^* = (k^*)^\alpha
\]
so that
\[
y^* = \left( \frac{s}{g + n + \delta} \right)^{\frac{\alpha}{1-\alpha}}.
\]
Taking logs we have
\[
\log y^* = \frac{\alpha}{1-\alpha} \left[ \log s - \log (g + n + \delta) \right].
\]
Taking now derivatives, we have
\[
\frac{d \log y^*}{d \log s} = \frac{\alpha}{1-\alpha}
\]
Note that, generally speaking,
\[
\frac{d \log y^*}{d \log s} = \frac{\frac{dy^*}{y^*}}{\frac{ds}{s}} = \frac{\Delta%y^*}{\Delta%s}
\]
so that the elasticity of output with respect to savings is
\[
\frac{\Delta%y^*}{\Delta%s} = \frac{\alpha}{1-\alpha}
\]
If \( \alpha = \frac{1}{3} \) (which makes sense empirically), this result above implies that a 10% increase in savings leads to a 5% increase in output per capita.

### 4.5 Summary

The Solow growth model implies a transition equation for capital per unit of effective labor of the form
\[
\frac{dk(t)}{dt} = sf(k(t)) - (g + n + \delta) k(t).
\]
which has a unique steady-state that satisfies
\[
sf(k^*) = (g + n + \delta) k^*
\]
and stable dynamics around this steady state—at \( k \) below \( k^* \), \( k \) is increasing, and at \( k \) above \( k^* \), \( k \) is decreasing.

Increases in the savings rate lead to a permanent increase in the level of capital per effective unit of labor, and hence output per effective unit of labor. Measuring living standards as output per worker

\[
\frac{Y(t)}{L(t)} = A(t) f(k(t))
\]

the steady-state level of output per worker is

\[
\frac{Y(t)^*}{L(t)} = A(t) f(k^*)
\]

while the growth rate of output per worker in steady-state is

\[
\frac{d \left( \log \frac{Y(t)^*}{L(t)} \right)}{dt} = \frac{d \log A(t)}{dt} = g
\]

A rise in the savings rate leads to a permanent increase in output per worker but does not influence the long-run growth rate.

5 Consumption

In the Solow growth model consumption is equal to output minus investment:

\[
C(t) = Y(t) - I(t).
\]

Since investment is proportional to output, we have

\[
C(t) = (1 - s) Y(t).
\]

Let \( c(t) = C(t)/(A(t)L(t)) \) denote consumption per effective unit of labor. Dividing through by effective units of labor we have

\[
c(t) = (1 - s) y(t).
\]

The effect of an increase in savings has an ambiguous effect on consumption. To see this, note that, holding \( y(t) \) fixed, an increase in the savings rate would reduce consumption. The increase in the savings rate also increase output \( y(t) \) however. In the short-run, we expect that the effect on output is small and an increase in the savings rate would reduce consumption at least temporarily.
In the long-run, the effect is ambiguous however and would depend on the actual savings rate—at low savings rates, an increase in savings would raise consumption, while at high savings rates, an increase in savings rates would reduce consumption.

To understand the long-run effects of a change in the savings rate on household well-being, let’s consider the effect of a change in savings on the steady-state level of consumption per effective unit of labor:

\[ c^* = (1 - s) y^*. \]

Taking derivatives we have:

\[ \frac{dc^*}{ds} = -y^* + (1 - s) \frac{dy^*}{ds}. \]

From our solution for \( y^* \) we can compute

\[ \frac{dy^*}{ds} = \frac{\alpha}{1 - \alpha} \frac{y^*}{s} \]

so that

\[ \frac{dc^*}{ds} = \left( \frac{\alpha (1 - s)}{s (1 - \alpha)} - 1 \right) y^* \]
\[ = \frac{\alpha - s}{(1 - \alpha) s} y^*. \]

This implies that

\[ \frac{dc^*}{ds} > 0 \text{ if } s < \alpha, \]
\[ \frac{dc^*}{ds} < 0 \text{ if } s > \alpha. \]

These results show that, for low levels of the savings rate, an increase in savings will raise consumption and therefore raise the long-run level of household well-being. At high levels of the savings rates, an increase in the savings rate would reduce consumption and therefore reduce the long-run level of household well-being.

## 6 Convergence

An important question for growth is whether or not poor countries will catch up to rich countries through more rapid growth. To study this question, consider the percentage change in \( k_t \):

\[ \frac{d \log k(t)}{dt} = \frac{dk(t)}{dt} \frac{1}{k(t)}. \]
From the transition equation for $k(t)$ we have

$$\frac{d \log k(t)}{dt} = s \frac{f(k(t))}{k(t)} - (g + n + \delta)$$

Since $f(k)/k$ is decreasing in $k$, the growth rate of $k(t)$ is high when $k(t)$ is low.

**Unconditional convergence.** For two countries with identical technologies, savings rates, production structures and population growth rates, the Solow model implies that the poorer country—i.e. the country with a lower initial level of capital and hence output per capita will grow at a faster rate than the rich country. Over time, the living standard of the poor country will catch up to that of the rich country. The living standards of these two economies will converge to the same level. This is often called unconditional convergence in the growth literature.

**Conditional convergence.** Now consider two countries with different initial levels of technology but the same savings rates, production structures and population growth rates. Furthermore, suppose that each country’s technology grows at the same rate, so that one country has a permanently higher level of technology than another country. In this case, although neither country may start out at the steady-state, both economies will eventually converge to steady states with the same level of capital per effective unit of labor $k^*$ and, in the long-run therefore will have the same levels of output per effective unit of labor $y^*$. The gap between the rich (technologically advanced) economy and the poor (technologically behind) will remain, and will reflect the differences in the initial levels of technology.

Conditional on this technology gap, these countries will in the long-run experience the same growth rate. Similarly, if one country saves more than another country on a permanent basis, these countries will also experience the same growth rate in the long-run but will exhibit permanent differences in the level of income per capita. Such convergence in growth rates is often referred to as “conditional convergence”.

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7 Implications of the Solow Growth Model

7.1 Living standards in the long run

• The Solow Growth Model implies that the economy will achieve a balanced growth path where GDP per capita grows at a constant rate equal to the rate of technological change.

• An increase in the savings rate will raise the level of GDP per capita.

• An increase in the population growth rate will lower the level of GDP per capita.

• Changes in population growth or savings rate will not affect the long-run growth of GDP per capita.

7.2 Convergence

• If all countries have access to the same technology $A_t$ and the same savings rate, the Solow model implies convergence in per-capital incomes $Y_t/L_t$, i.e. everything else equal, the Solow model implies that a poor country should grow faster than a rich country and catch up.

• If countries differ in terms of the initial level of technology or savings rates they will not converge to the same per capita income levels.

• If the rate of technological growth is the same across countries, the Solow growth model implies that the growth rates of income per capita should converge even if the levels do not.

7.3 Some empirical findings

• The data imply that countries have not been converging in per capita income terms.

• Controlling for savings rates and other long-run determinants of income, one tends to find that initial income is negatively correlated with growth rates, implying “conditional convergence”.

• Mankiw-Romer-Weil argue that an augmented version of the Solow model fits the data very well in terms of convergence properties across countries (i.e. poor countries do tend to grow faster than rich ones in a manner similar to that described by model dynamics). The
augmented model allows for human as well as physical capital accumulation. The authors argue that the true capital share should incorporate human as well as physical capital, in which case $\alpha K_t$ is better measured as $\frac{2}{3}$ rather than $\frac{1}{3}$. With a higher capital share, the model converges more slowly and is better able to fit the facts.

8 Growth Accounting

The neoclassical production structure embedded in the Solow Growth Model provides a convenient accounting framework which one may use to decompose the various sources of growth. We begin by presenting the growth accounting formula. To use this formula we need to add some assumptions regarding markets and market structure.

8.1 Growth Accounting formula

In this section we obtain the formula for growth accounting based on the Solow Model. For this we come back to discrete time because the data necessary for this procedure (mainly coming from National Accounting) has commonly quarterly frequency.

Let’s start with the familiar Cobb-Douglas case:

$$Y_t = K_t^\alpha (A_t L_t)^{1-\alpha}$$

Taking logs we have

$$\log Y_t = \alpha \log K_t + (1-\alpha) \log L_t + (1-\alpha) \log A_t$$

Similarly, if we took logs of the production function at time $t-1$ we would have:

$$\log Y_{t-1} = \alpha \log K_{t-1} + (1-\alpha) \log L_{t-1} + (1-\alpha) \log A_{t-1}$$

Taking the difference between these two equations,

$$\Delta \log Y_t = \alpha \Delta \log K_t + (1-\alpha) \Delta \log L_t + (1-\alpha) \Delta \log A_t$$

Note that for any variable $X$

$$\Delta \log X_t = \log (X_t) - \log (X_{t-1}) = \log \left( \frac{X_t}{X_{t-1}} \right)$$
\[ \log \left( \frac{X_{t-1} + \Delta X_t}{X_{t-1}} \right) = \log (1 + g_X) \approx g_X \]

for \( g_X \) small. Therefore, equation (2) may be represented as

\[ g_Y = \alpha g_K + (1 - \alpha) g_L + (1 - \alpha) g_A. \]

This equation states that the growth rate of output is a weighted average of the growth rates of capital, labor and technology. The weights depend on the coefficient \( \alpha \) in the production function.

To use this formula, we thus need to be able to measure \( \alpha \). To do so, we consider the firm’s problem.

### 8.2 Firm’s problem

We assume that firms in the economy hire labor and capital as inputs to the production function

\[ Y_t = F (K_t, A_t, L_t). \]

Firms hire labor at the real wage rate \( W_t \) and rent capital at rate \( R^K_t \). The economy is characterized by perfect competition so that firms are price takers in both input and output markets. We assume that the output price is unity. Thus, \( Y_t \) may be interpreted as real units of output.

Profits of a firm that hires inputs \( K_t \) and \( L_t \) are

\[ \pi_t = Y_t - W_t L_t - R^K_t K_t. \]

To maximize profits, the firm should hire an additional input up until the point where the marginal benefit of the additional input equals its marginal cost. The marginal benefit is the extra unit of output obtained from hiring one more unit of an input. The marginal cost is the market price of hiring an additional unit of an input, i.e., the wage and the rental rate of capital.

The capital input is chosen to equate the marginal product of capital with the rental rate on capital goods.

\[ MPK_t = R^K_t \]

where \( MPK_t \) denotes the marginal product of capital at time \( t \).
The labor input is chosen to equate the marginal product of labor with the real wage:

\[ MPL_t = W_t \]

With Cobb-Douglas production, these conditions become

\[
MPK_t = \frac{\alpha Y_t}{K_t} = R_t^K,
\]

\[
MPL_t = (1 - \alpha) \frac{Y_t}{L_t} = W_t.
\]

Using these equations, we get

\[
\alpha = \frac{R_t^K K_t}{Y_t},
\]

\[
1 - \alpha = \frac{W_t L_t}{Y_t}.
\]

### 8.3 Factor shares and output elasticities

This last equation implies that the value \(1 - \alpha\) is equal to the share of output that is paid out in wages. To measure \(1 - \alpha\) in the data we would compute total labor compensation as a fraction of GDP. We compute labor share of output as

\[
1 - \alpha = \frac{W_t L_t}{Y_t} = \frac{\text{total labor compensation}}{\text{GDP}}.
\]

Applying the first-order condition for labor above we have

\[
1 - \alpha = MPL_t \frac{L_t}{Y_t} = \frac{\Delta Y_t}{\Delta L_t} \frac{L_t}{Y_t} = \frac{\Delta Y_t / Y_t}{\Delta L_t / L_t}.
\]

The right hand side of this expression above is interpreted as the percent change in output obtained from a one percent change in labor, i.e. the elasticity of output with respect to labor. Thus the value \(1 - \alpha\) is both the share of output going to labor in the form of compensation and the elasticity of output with respect to labor.

In summary, with perfect competition, labor share equals the elasticity of output with respect to labor. In the U.S., direct estimates of the wage bill and GDP allows us to estimate labor share. It is approximately 0.64 for the US (and in the same range for Europe) and relatively stable over the last 100 years.
Capital share is similarly defined as

\[ \alpha = \frac{R^K_t K_t}{Y_t} = \frac{\Delta Y_t / Y_t}{\Delta K_t / K_t}. \]

In principle, we could compute capital share directly from the data and obtain an independent estimate of \( \alpha \) to corroborate with the estimate of \( 1 - \alpha \) obtained from the labor compensation data. Because both the capital stock and the return on capital are hard to measure, capital share is more difficult to estimate directly from the data however. Under the assumptions of perfect competition and constant-returns to scale, we know that \( \pi_t = 0 \) (firms earn zero economic profits) and therefore

\[ Y_t = W_t L_t + R^K_t K_t \]

so that

\[ \frac{R^K_t K_t}{Y_t} = 1 - \frac{W_t L_t}{Y_t} \]

This implies that capital’s share \( \frac{R^K_t K_t}{Y_t} = .36 \). Note that this calculation does not explicitly rely on a Cobb-Douglas production function–given labor share, we can obtain an estimate of capital’s share as long as we are willing to assume that the economy is reasonably well characterized by constant returns to scale technology and perfect competition.

### 8.4 Summary of growth accounting

Our growth accounting identity implies

\[ g_Y = \alpha g_K + (1 - \alpha) g_L + (1 - \alpha) g_A. \]

Thus, the sources of growth in output can be attributed to growth in factor inputs, weighted by their factor share, and the effect of growth in technology through the term \( (1 - \alpha) g_A \). Note that \( A \) is often referred to as Total Factor Productivity (TFP).

To interpret this equation, consider a 1% increase in capital. The elasticity of output with respect to capital is \( \alpha \). Hence, a 1% increase in capital leads to an .36% increase in output, given our estimate of \( \alpha = .36 \).

To make this operational as an accounting identity, we measure growth in capital and labor, along with growth in output from data on capital, labor hours and real GDP. The contribution of
“technology” measured by \((1 - \alpha) g_A\) is thus computed as a “Solow” residual (i.e. how much of output growth cannot be explained by growth in factor inputs given measured shares).

### 8.5 Some facts

For the $$1947-1973$$ period, the U.S. experienced GDP growth of 4.02%. Growth from capital accounted for 1.71% and growth from labor accounted for 0.95%. TFP growth was 1.35% which accounted for 33.6% of growth in GDP per capita over this period.

In contrast, Germany experienced 6.61% GDP growth over this period, 56% of this growth was accounted for by TFP (and hence not explained by labor force or catch-up through capital accumulation). Thus, the German economic miracle appears to have more to do with adoption of technology (growth in \(A\)) than catch-up in terms of capital accumulation.

Alwyn Young argues that, contrary to past beliefs, the economic miracles of South-East Asia had more to do with accumulation of physical and human capital, and less to do with technological change (growth in TFP).

#### 8.5.1 Real wages

Profit maximization by firms in a Cobb-Douglas production framework implies that the real wage is proportional to labor productivity. Growth in labor productivity should therefore lead to growth in real wages and increases in labor productivity are fully reflected in worker wages. To see whether this is true in U.S. data, we calculate the real wage as the nominal wage/CPI. The following table summarizes some key findings regarding wage and productivity growth.

<table>
<thead>
<tr>
<th></th>
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<th></th>
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</thead>
<tbody>
<tr>
<td>Real Wage</td>
<td>2.6%</td>
<td>.7%</td>
<td>3.8%</td>
</tr>
<tr>
<td>Labor Productivity</td>
<td>2.7%</td>
<td>1.5%</td>
<td>2.9%</td>
</tr>
</tbody>
</table>

First, real wages grew steadily in the $$1960-1973$$ period and then slowed down in the $$1974-1995$$ period. Labor productivity grew steadily in the $$1960-1973$$ period and then slowed down as well. The slowdown in real wages was larger than the slowdown in labor productivity however, which is a puzzle for a simple model of homogeneous labor and Cobb-Douglas production.
8.5.2 The contribution of IT capital during the high-tech boom of the late 1990’s

The next table provides a decomposition of the sources of output growth over the 1973 – 2000 period.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>TFP</td>
<td>.35%</td>
<td>1.4%</td>
</tr>
<tr>
<td>Labor</td>
<td>.20%</td>
<td>.20%</td>
</tr>
<tr>
<td>Other capital</td>
<td>.30%</td>
<td>.10%</td>
</tr>
<tr>
<td>IT capital</td>
<td>.45%</td>
<td>1.0%</td>
</tr>
<tr>
<td>TOTAL</td>
<td>1.3%</td>
<td>2.7%</td>
</tr>
</tbody>
</table>

This decomposition considers the separate effects of IT (information technology) capital, and non-IT capital. The growth accounting framework assumes a Cobb-Douglas production function with both IT and non-IT capital:

\[
g_Y = \alpha_{IT} g^I_T + \alpha_{NIT} g^N_T + (1 - \alpha_{IT} - \alpha_{NIT}) g_L + (1 - \alpha_{IT} - \alpha_{NIT}) g_A.\]

For the period 1973 − 2000, \( \alpha_{IT} = .05 \) provides a reasonable benchmark for the IT capital share. Assuming that labor’s share is fixed at .64, this implies a non-IT capital share \( \alpha_{NIT} = .31 \) so that the total capital share is again .36. To highlight the role that IT capital played in recent growth accounting, the sample is divided into the 1973 − 1995 period and the 1996 − 2000 period. The major findings are:

1. During the 1973 − 1995 period, one half of output growth is due to capital accumulation (IT and non-IT combined), the rest of output growth is due to labor and TFP growth.

2. In contrast, during the 1996 − 2000, half of output growth is due to TFP growth and a large fraction of the rest of the output growth is due to growth in IT capital alone.

3. The role of other both other capital and labor growth is negligible during this episode.

Thus, one way to understand what happened to the U.S. economy during the 1990’s may be characterized as a boom in technology that led to rapid growth in both TFP and IT capital, both of which were reflected in labor productivity and output growth.
8.5.3 The growth of East-Asian dragons

Alwyn Young (1995) documents the fundamental role played by factor accumulation in explaining the extraordinary postwar growth of Hong Kong, Singapore, South Korea, and Taiwan. Participation rates, educational levels, and (excepting Hong Kong) investment rates have risen rapidly in all four economies. In addition, in most cases there has been a large intersectoral transfer of labor into manufacturing, which has helped fuel growth in that sector. Once one accounts for the dramatic rise in factor inputs, one arrives at estimated total factor productivity growth rates that are closely approximated by the historical performance of many of the OECD and Latin American economies. While the growth of output and manufacturing exports in the newly industrializing countries of East Asia is virtually unprecedented, the growth of total factor productivity in these economies is not.

A proper measurement of factors changes our interpretation of that period.

<table>
<thead>
<tr>
<th></th>
<th>GDP growth</th>
<th>TFP growth</th>
<th>period</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hong-Kong</td>
<td>7.3%</td>
<td>2.3%</td>
<td>1966-1991</td>
</tr>
<tr>
<td>Singapore</td>
<td>8.7%</td>
<td>0.2%</td>
<td>1966-1990</td>
</tr>
<tr>
<td>South-Korea</td>
<td>8.5%</td>
<td>1.7%</td>
<td>1966-1990</td>
</tr>
<tr>
<td>Taiwan</td>
<td>10.8%</td>
<td>2.1%</td>
<td>1966-1990</td>
</tr>
<tr>
<td>France</td>
<td>1.2%</td>
<td>1.5%</td>
<td>1960-1989</td>
</tr>
<tr>
<td>Italy</td>
<td>-</td>
<td>2%</td>
<td>1960-1989</td>
</tr>
<tr>
<td>Venezuela</td>
<td>-</td>
<td>2.6%</td>
<td>1960-1989</td>
</tr>
<tr>
<td>UK</td>
<td>1.3%</td>
<td>1.3%</td>
<td>1960-1989</td>
</tr>
<tr>
<td>US</td>
<td>2.8%</td>
<td>0.4%</td>
<td>1960-1989</td>
</tr>
</tbody>
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